Notes on Rational Inequalities

To Solve Rational Inequalities:

1. Write the inequality as an equation
2. Solve the equation
3. Determine all values that make the denominator zero
4. Draw a number line, and mark all the solutions and critical values from steps 2 and 3
5. Select a test point in each interval between the marks on the number line, and determine if they satisfy the original inequality
6. If a test point satisfies the inequality, then every point on that interval does so also, and that interval should be identified in the solution
7. State your answer including all intervals that solve the inequality

EXAMPLES

Solving a rational inequality: Problem type 1

Solve the following inequality.

$$\frac{x-7}{x-2} \leq 0$$

Write your answer using interval notation.

We need to find all the values of x that make the quotient $\frac{x-7}{x-2}$ *negative or equal to 0.*

To find these values, we do a sign analysis.

We first look at the sign of the numerator $x-7$.

If $x=7$, then $x-7$ is 0.
If $x<7$, then $x-7$ is negative.
If $x>7$, then $x-7$ is positive.

We show this on the number line.

	7
sign of $x-7$:	− − − − − − − − − − − − − − − **0** + + + + + + +

We then look at the sign of the denominator $x-2$.

If $x=2$, then $x-2$ is 0.
If $x<2$, then $x-2$ is negative.
If $x>2$, then $x-2$ is positive.

	2 7
sign of $x-7$:	− − − − − − − − − − − − − − − **0** + + + + + + +
sign of $x-2$:	− − − − − − − **0** + + + + + + + + + + + + + + +

Finally, to find the sign of $\frac{x-7}{x-2}$, we use the rules for dividing signed numbers.

	2 7
sign of $x-7$:	− − − − − − − − − − − − − − − **0** + + + + + + +
sign of $x-2$:	− − − − − − − **0** + + + + + + + + + + + + + + +
	positive quotient ↓ undefined negative quotient ↓ 0 positive quotient

So, we have $\frac{x-7}{x-2} \leq 0$ when $2<x$ and $x \leq 7$.

Note that for $x=2$, the quotient is undefined and so 2 *is not* part of the solution.
For $x=7$, the quotient is 0 and so 7 *is* part of the solution.
We write the solution in interval notation.

$\boxed{(2,7]}$

135 NOTES ON RATIONAL INEQUALITIES

Solving a rational inequality: Problem type 1

Solve the following inequality.

$$\frac{x+1}{-x-6} > 0$$

Write your answer using interval notation.

We need to find all the values of x that make the quotient $\frac{x+1}{-x-6}$ *positive*.

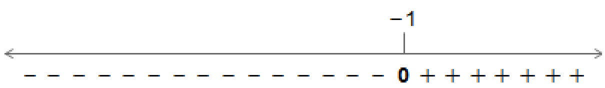
To find these values, we do a sign analysis.

We first look at the sign of the numerator $x+1$.

If $x = -1$, then $x+1$ is 0 .
If $x < -1$, then $x+1$ is negative.
If $x > -1$, then $x+1$ is positive.

We show this on the number line.

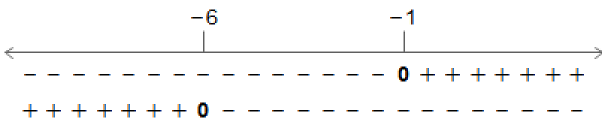
sign of $x + 1$:



We then look at the sign of the denominator $-x-6$.

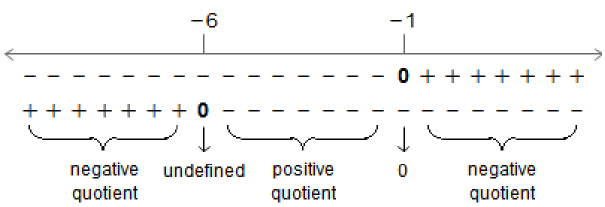
If $x = -6$, then $-x-6$ is 0 .
If $x < -6$, then $-x-6$ is positive.
If $x > -6$, then $-x-6$ is negative.

sign of $x + 1$:
sign of $-x - 6$:



Finally, to find the sign of $\frac{x+1}{-x-6}$, we use the rules for dividing signed numbers.

sign of $x + 1$:
sign of $-x - 6$:



So, we have $\frac{x+1}{-x-6} > 0$ when $-6 < x$ and $x < -1$.

Note that for $x = -6$, the quotient is undefined and so -6 *is not* part of the solution.
For $x = -1$, the quotient is 0 and so -1 *is not* part of the solution.
We write the solution in interval notation.

$$\boxed{(-6, -1)}$$

135 NOTES ON RATIONAL INEQUALITIES

Solving a rational inequality: Problem type 2

Solve the following inequality.

$$\frac{x-7}{x+5} \geq -3$$

Write your answer as an interval or union of intervals.

If there is no real solution, click on "No solution".

We first rewrite the inequality so that 0 is on one side.

$$\frac{x-7}{x+5}+3 \geq 0$$

We first rewrite the inequality so that 0 is on one side.

$$\frac{x-7}{x+5}+3 \geq 0$$

Then, we find a common denominator and simplify.

$$\frac{x-7+3(x+5)}{x+5} \geq 0$$
$$\frac{x-7+3x+15}{x+5} \geq 0$$
$$\frac{4x+8}{x+5} \geq 0$$

A rational expression can only change signs at the x-values that make its numerator or denominator zero.

For the rational expression $\frac{4x+8}{x+5}$, we get the following.

Zero(s) of the numerator:

$$\begin{aligned} 4x+8 &= 0 \\ 4x &= -8 \\ x &= -2 \end{aligned}$$

Zero(s) of the denominator:

$$\begin{aligned} x+5 &= 0 \\ x &= -5 \end{aligned}$$

The values $x=-5$ and $x=-2$ split the real numbers into the following intervals.

$$(-\infty,-5),\ (-5,-2),\ (-2,\infty)$$

For each interval, the rational expression will always be positive or always be negative.
So, we choose a test value in each interval and evaluate the rational expression at that value.
The sign of the rational expression at the test value tells us the sign for the entire interval.

Interval	Test value (x)	Value of $\frac{4x+8}{x+5}$
$(-\infty,-5)$	-6	$\frac{4(-6)+8}{-6+5} > 0$
$(-5,-2)$	-3	$\frac{4(-3)+8}{-3+5} < 0$
$(-2,\infty)$	0	$\frac{4(0)+8}{0+5} > 0$

We are trying to solve the inequality $\frac{4x+8}{x+5} \geq 0$.

So, the solution is the set of all x-values that make the rational expression ***positive or*** 0.

From the table above, we see that the rational expression is positive on the following intervals.

$$(-\infty,-5)$$
$$(-2,\infty)$$

A rational expression equals 0 when its numerator equals 0 and its denominator does not.
So, our rational expression equals 0 at $x=-2$.

Including $x=-2$ with the intervals above gives the answer.

$$\boxed{(-\infty,-5)\cup[-2,\infty)}$$

135 NOTES ON RATIONAL INEQUALITIES

Solving a rational inequality: Problem type 2

Solve the following inequality.

$$\frac{7-x}{x+1} > \frac{4-x}{x+3}$$

Write your answer as an interval or union of intervals.

If there is no real solution, click on "No solution".

We first rewrite the inequality so that 0 is on one side.

$$\frac{7-x}{x+1} - \frac{4-x}{x+3} > 0$$

Then, we find a common denominator and simplify.

$$\frac{(7-x)(x+3)-(4-x)(x+1)}{(x+1)(x+3)} > 0$$

$$\frac{7x+21-x^2-3x-\left(4x+4-x^2-x\right)}{(x+1)(x+3)} > 0$$

$$\frac{21+4x-x^2-\left(4+3x-x^2\right)}{(x+1)(x+3)} > 0$$

$$\frac{21+4x-x^2-4-3x+x^2}{(x+1)(x+3)} > 0$$

$$\frac{x+17}{(x+1)(x+3)} > 0$$

A rational expression can only change signs at the x-values that make its numerator or denominato

For the rational expression $\frac{x+17}{(x+1)(x+3)}$, we get the following.

Zero(s) of the numerator:

$$x+17 = 0$$
$$x = -17$$

Zero(s) of the denominator:

$$(x+1)(x+3) = 0$$
$$x+1 = 0 \quad \text{or} \quad x+3 = 0$$
$$x = -1 \qquad\quad x = -3$$

The values $x=-17$, $x=-3$, and $x=-1$ split the real numbers into the following intervals.

$(-\infty,-17)$, $(-17,-3)$, $(-3,-1)$, $(-1,\infty)$

For each interval, the rational expression will always be positive or always be negative.
So, we choose a test value in each interval and evaluate the rational expression at that value.
The sign of the rational expression at the test value tells us the sign for the entire interval.

Interval	Test value (x)	Value of $\frac{x+17}{(x+1)(x+3)}$
$(-\infty,-17)$	-18	$\frac{-18+17}{(-18+1)(-18+3)} < 0$
$(-17,-3)$	-4	$\frac{-4+17}{(-4+1)(-4+3)} > 0$
$(-3,-1)$	-2	$\frac{-2+17}{(-2+1)(-2+3)} < 0$
$(-1,\infty)$	0	$\frac{0+17}{(0+1)(0+3)} > 0$

We are trying to solve the inequality $\frac{x+17}{(x+1)(x+3)} > 0$.

So, the solution is the set of all x-values that make the rational expression ***positive***.

From the table above, we see that the rational expression is positive on the following intervals.

$(-17,-3)$
$(-1,\infty)$

Here is the answer.

$$\boxed{(-17,-3)\cup(-1,\infty)}$$