Agenda

MIDTERM: Honorlock

Review Quiz 6

Optimization and related rates

Lecture Logarithmic Differentiation

Review Analysis of Functions with Regressions

Project Find Critical Points and Inflection points

Review Quiz 6

Determine the intervals where the graph of the given function is concave up and concave down.

f'

$f(x) = 4x^{4/3} + 18x^{1/3}$

Concave up: $x >$ 9/4 and $x <$ 0, concave down: 0 $< x <$ 9/4

$f'(x) = \frac{16}{3}x^{1/3} + 6x^{-2/3}$

$f''(x) = \frac{16}{9}x^{-2/3} - 4x^{-5/3} < 0$ Concave down

$x^{-5/3}\left(\frac{16}{9}x - 4\right) < 0$

$x^{-5/3} = 0 \quad x = 0$

$\frac{16}{9}x - 4 = 0 \quad \frac{4 \cdot 9}{16} = \frac{9}{4}$

$x^{-3} - x^{-2} = 0$

$x^{-3} \cdot x = x^{-2}$

$x^{-3}(1 - x^{1})$

0, 9/4

$$x^2 - 5x^1 = x^1(x-5)$$

$$x^5 - 2x^7 = x^5(1-2x^2)$$

$$x^{-7} - x^4 = x^{-7}(1-x^{11})$$

$$x^{-3/5} - x^{-7/5} = x^{-7/5}\left(x^{\boxed{4/5}} - 1\right)$$

Determine the intervals where the graph of the given function is concave up and concave down.

$f(x) = 4x^{4/3} + 18x^{1/3}$

Concave up: $x >$ 9/4 and $x <$ 0, concave down: 0 $< x <$ 9/4

$y' = \frac{16}{3}x^{1/3} + 6x^{-2/3}$

$y'' = \frac{16}{9}x^{-2/3} - 4x^{-5/3}$

$0 = \underline{\underline{x^{-5/3}}}\left(\frac{16}{9}x - 4\right)$

$\frac{16}{9}x - 4 = 0 \qquad x = \frac{36}{16} = 9/4$

$\leftarrow 0 \text{ --- } 9/4 \rightarrow$

$y''(1) = \frac{16}{9} - 4 = \frac{16}{9} - \frac{36}{9} < 0$

CONCAVE DOWN

at $x = 1$

$$y = 4x^{4/3} + 8x^{1/3}$$

$$y' = \frac{16}{3}x^{1/3} + \frac{8}{3}x^{-2/3}$$

$$y'' = \frac{16}{9}x^{-2/3} - \frac{16}{9}x^{-5/3}$$

$$= x^{-5/3}\left(\frac{16}{9}x - \frac{16}{9}\right)$$

$$= 0 \qquad\qquad = 1$$

Your response:

A function f has a slant asymptote $y = mx + b$ $(m \neq 0)$ if $\lim\limits_{x \to \infty} [f(x) - mx + b] = 0$

and/or $\lim\limits_{x \to -\infty} [f(x) - mx + b] = 0$.

Find the slant asymptote. (Use long division to rewrite the function.) Then, graph the function and its asymptote on the same axes.

$$f(x) = \frac{x^4}{x^3 + 1}$$

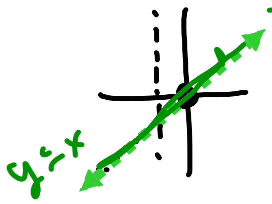
$1\frac{1}{2}$

$x - \frac{x}{x^3+1}$

$x^3+1 \overline{) \; x^4 + 0x^3 + 0x^2 + 0x + 0}$

$-(x^4 + x)$

$-x$



Your response:

A function f has a slant asymptote $y = mx + b$ ($m \neq 0$) if $\lim_{x \to \infty} [f(x) - mx + b] = 0$ and/or $\lim_{x \to -\infty} [f(x) - mx + b] = 0$.

Find the slant asymptote. (Use long division to rewrite the function.) Then, graph the function and its asymptote on the same axes.

$$f(x) = \frac{x^4}{x^3 + 1}$$

$$x - \frac{x}{x^3+1}$$

$$x^3 + 1 \;\big)\; x^4 + 0x^3 + 0x^2 + 0x$$

$$-(x^4 \qquad\qquad\qquad + x)$$

$$-x$$

$$= \boxed{\frac{x^4}{x^3+1} = x - \frac{x}{x^3+1}}$$

Ex $\dfrac{x^4 + x^3}{x^3+1}$

$$x + 1$$

$$x^3 + 1 \;\big)\; x^4 + x^3$$

$$x^4 \qquad + x$$

$$x^3$$

Slant

$y = x + 1$

Determine where the graph of $f(x) = 2x^3 + 15x^2 + 10x - 82$ is concave down.

$f(x)$ is concave down where x $\boxed{< \frac{-30}{12}}$.

$f' = 6x^2 + 30x + 10$

$f'' = 12x + 30 < 0$

$12x < -30$

$x < -\frac{30}{12}$

Determine where the graph of $f(x) = 2x^3 + 15x^2 + 10x - 82$ is concave down.

$f(x)$ is concave down where x $\boxed{<}$ $-\frac{30}{12}$.

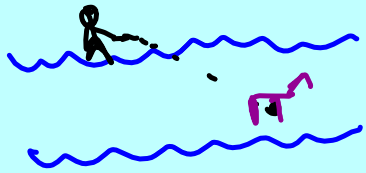
$y' = 6x^2 + 30x + 10$

$y'' = 12x + 30 < 0$

$12x < -30$

$x < -\frac{30}{12} =$

Lecture: Optimization



MAX/MIN

$f'(\text{MAX}) = 0$ $\qquad$ $f'(\text{MIN}) = 0$

$x =$ $\qquad$ $x =$

$f''(\text{MAX}) = -$ $\qquad$ $f''(\text{min}) = +$

$x =$

down $\qquad$ up

1. Find function to optimize.
2. Find critical points

   $f'(x) = 0$ or undefined
3. Find f''(critical points)

   determine if max or min

Ex $\boxed{f(x) = x^2 - 6x + 7}$

$f'(x) = 2x - 6 = 0 \rightarrow x = 3$

critical where

$f'(3) = 2(3) - 6 = 0$

$f''(x) = 2$

$f''(3) = 2$ + UP

$f(3) = \min$

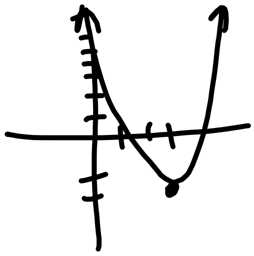
$f(3) = 9 - 18 + 7 = -2$

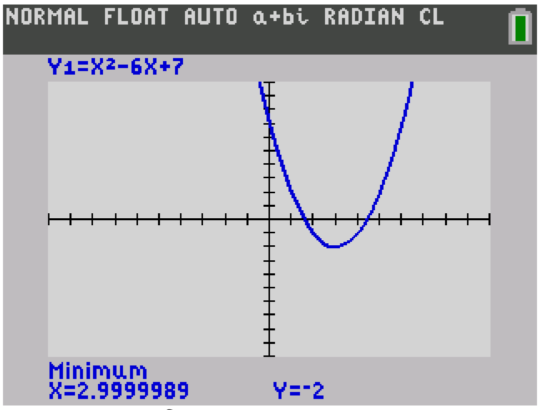
$f(0) = 7$

$f(1) = 0$

$f(10) = 47$

$y = x^2 - 6x + 7$





$3 \approx$ ~

Optimization (MAX / MINS)

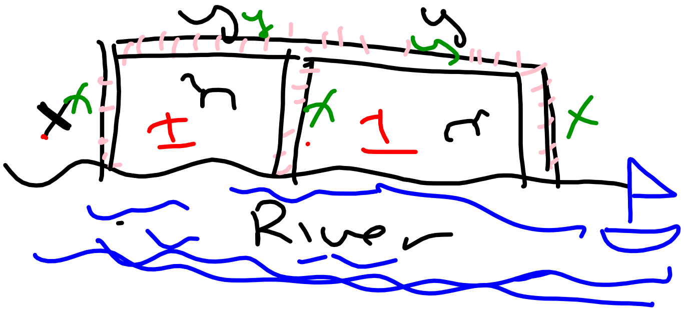
Main Idea $f(x, y)$

Constraint (equation relates x & y)

Multiv-variable function

Grade (Roll, Mood)

Ex



1000' of Fence.

$\boxed{\text{Maximize Area}}$ (Main Idea)

$A(x,y) = 2xy$

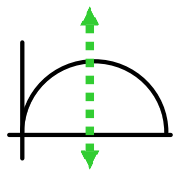
Constraint $\boxed{1000' = 3x + 2y}$

$2y = 1000 - 3x$

$y = \dfrac{1000 - 3x}{2}$

$A(x) = 2x\left(\dfrac{1000-3x}{2}\right)$

$Y_1 = 2x(1000-3x)/2$



Window: Xmin 0
Xmax $333.\frac{1}{3}$

Calc 4: maximum
Left: 0
Right: 333

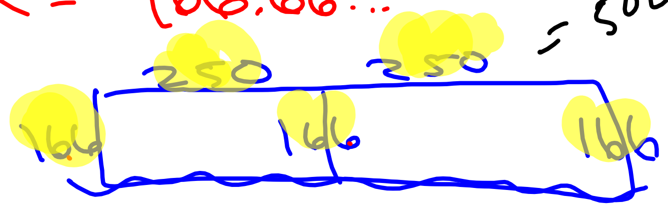
Constraint
$y = \dfrac{1000-3x}{2}$

$y = \dfrac{1000-500}{2}$

$= 250$

$x = 166.66\ldots$

$= 500$



$= x$ (500)

A = 83000

$A(x) = 2x\left(\frac{1000-3x}{2}\right) = 1000x - 3x^2$

$A' = 1000 - 6x = 0$ Find criticals

$6x = 1000$

$x = 166.\bar{6}$

$A'' = -6$

↑ down MAX

$y = \frac{1000-3x}{2} \to \frac{1000-3(166.\bar{6})}{2}$

$= \frac{500}{2} = 250$

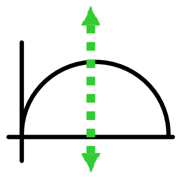
$A(x,y) = 2xy$

Constraint $\boxed{1000 = 3x + 2y}$

$2 \cdot 166.\bar{6} \cdot 250 = \frac{250000}{3} = 83,300$

$\frac{500}{3} \cdot 500$

$Y_1 = 2x(1000-3x)/2$



Window: Xmin 0

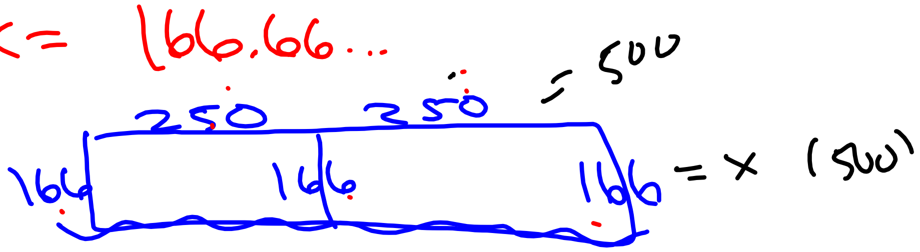
Xmax 333.1/3

Calc 4: maximum

Left: 0

Right: 333

$x = 166.66\ldots$



$$y = 2x\left(\frac{1000}{2} - \frac{3x}{2}\right)$$

$$\boxed{y = 1000x - 3x^2}$$

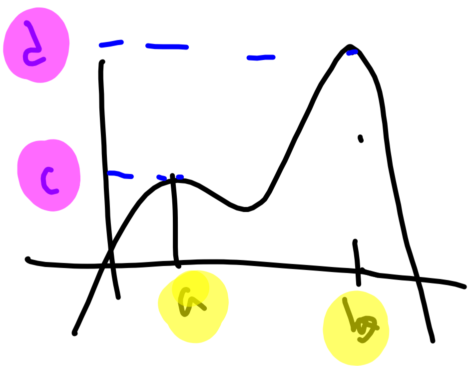
$$y' = 1000 - 6x = 0$$

$$6x = 1000$$

$$x = 166.66.$$

$$y = \frac{1000 - 3(166.66)}{2} = \frac{1000 - 500}{2} = 250$$

- What is max y-value
- when Where is max x-value



Where? a,b

What?

How to do by hand.

$A(x) = 1000x - 3x^2$

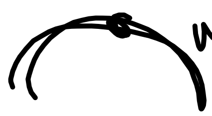
$A'(x) = 1000 - 6x = 0$ critical pts.

$1000 = 6x$

$x = 166\frac{2}{3}$

$A''(x) = -6$ concave down

$A''(166\frac{2}{3}) = -$

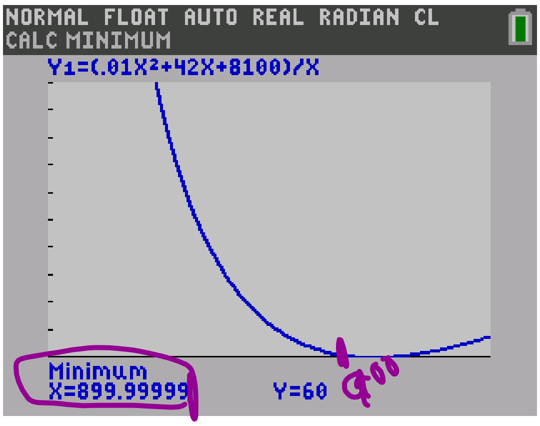


MAX.

Suppose that it costs a company $C(x) = 0.01x^2 + 42x + 8100$ dollars to manufacture x units of a product. For this cost function, the average cost function is $\overline{C}(x) = \frac{C(x)}{x}$.

Find the value of x that minimizes the average cost. The cost function can be related to the efficiency of the production process.

x = 900



min Ave cost.

$$\overline{C} = \frac{.01x^2 + 42x + 8100}{x}$$

$$\overline{C} = .01x + 42 + 8100x^{-1}$$

$$\overline{C}' = .01 - 8100x^{-2} = 0$$

$$.01 = \frac{8100}{x^2}$$

$$x^2 = 810000 \quad x = 900$$

Suppose that it costs a company $C(x) = 0.01x^2 + 42x + 8100$ dollars to manufacture x units of a product. For this cost function, the average cost function is $\overline{C}(x) = \dfrac{C(x)}{x}$.

Find the value of x that minimizes the average cost. The cost function can be related to the efficiency of the production process.

$x =$ [ ]

$\frac{12400}{100} = \$124$

$$\overline{C} = \frac{.01x^2 + 42x + 8100}{x}$$

$$\overline{C} = .01x + 42 + \frac{8100}{x} \quad (8100x^{-1})$$

1st Der.. $\overline{C}'$

$$\overline{C}' = .01 - 8100x^{-2} = 0$$

$$.01 = \frac{8100}{x^2}$$

$$x^2 = \frac{8100}{.01} = 810{,}000$$

$$x = 900$$

$$C'' = (2)8100x^{-3} \quad = C'' =$$

$$C''(900) = 2(8100)(900)^{-3} > 0 \quad \oplus$$

MINIMUM

Constraint: $100\ in^2$

Maximize Volume:

$$\text{Volume} = \pi r^2 h$$

$$(2)\ \text{Area Lid} = \pi r^2 + \pi r^2$$

$$\text{Side} = 2\pi r h$$

$$\text{Area} = \boxed{100 = 2\pi r^2 + 2\pi r h}$$

$$h = \frac{100 - 2\pi r^2}{2\pi r} = 4.6$$

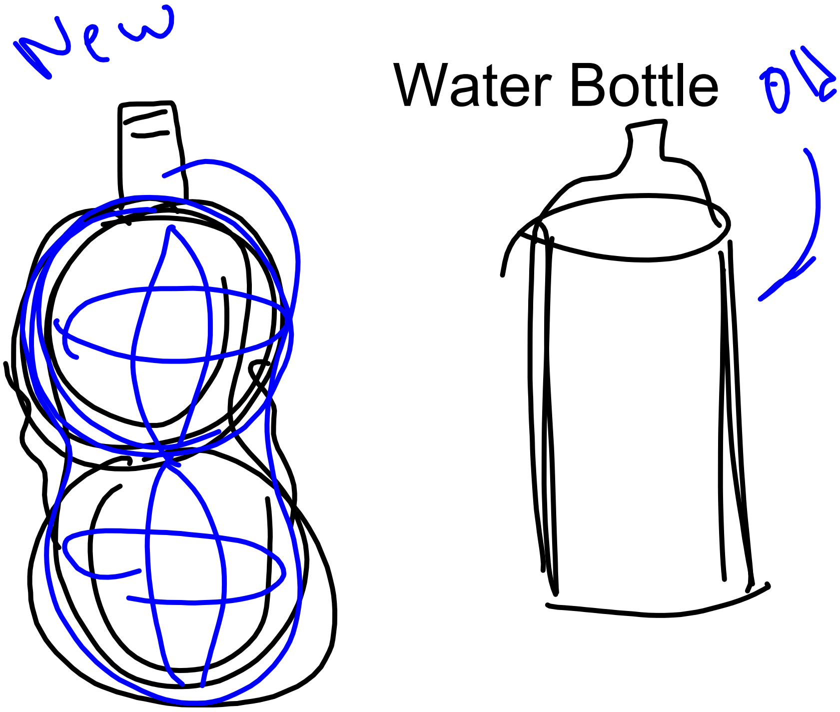
$$V(r,h) = \pi r^2 h$$

$$V(r) = \cancel{\pi} r^{\cancel{2}}\left(\frac{\overset{50}{\cancel{100}} - \cancel{2}\pi r^2}{\cancel{2\pi r}}\right) \qquad 50r - \pi r^3$$

$$Y_1 = x(100 - 2\pi x^2)/2 = 50x - \pi x^3$$

$$Y' = 50 - 3\pi x^2 = 0 \qquad x^2 = \frac{50}{3\pi} \qquad 2.3 = r$$

$$\sqrt{50/(3\pi)}$$


New
Water Bottle
Old

Related Rates

$$\left\{ \frac{dy}{dx} \right.$$

miles

Hours

mph

Rates

"per"

Balloon

If a spherical balloon is filling at 300 cc/sec dV/dt $\frac{\text{Volume}}{\text{time}}$. How fast is the radius changing at 5 cm? At 50 cm?

Looking for a Rate? dr/dt $\frac{\text{radius}}{\text{time}}$

Relate V and r

Sphere: $V = \frac{4}{3}\pi r^3$

$\frac{d}{dt}$ $\frac{d}{dt}$

Related Rates (Chain Rule)

$$\frac{dV}{dt} = \frac{4}{3}\pi\, 3r^2 \cdot \frac{dr}{dt}$$

$$300 = 4\pi (r)^2 \frac{dr}{dt}$$

$$\frac{dr}{dt} = \frac{300}{4\pi r^2}$$

$$\frac{dr}{dt}(r=5) = \frac{300}{4\cdot\pi\cdot 25}$$

$$\frac{dr}{dt}(r=50) = \frac{300}{4\pi\cdot 2500}$$

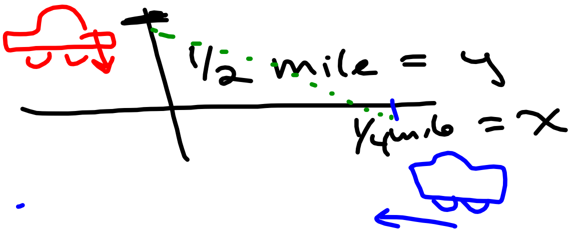
RATE Values

A car is traveling at 90 mph due south at a point 1/2 mile north of an intersection. A police car is traveling at 80 mph due west at a point 1/4 mile due east of the same intersection. At that instant, the radar in the police car measures the rate at which the distance between the two cars is changing. What does the radar gun register? Round your answer to three decimal places.

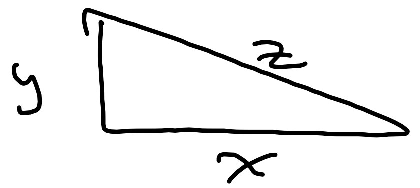
Your Answer:

Smaller

$-90 \text{ mph} = \frac{dy}{dt}$



$-80 = \frac{dx}{dt}$

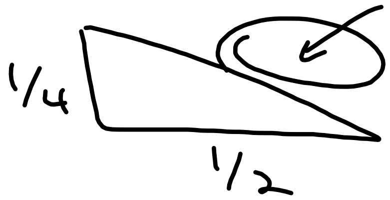


$z^2 = x^2 + y^2$

$\frac{d}{dt}(z^2)$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

Values		Rates	
$x =$	$1/4$	$dx/dt =$	-80
$y =$	$1/2$	$dy/dt =$	-90
$z =$	$1/\sqrt{20}$	$dz/dt =$	?



$\frac{1}{\sqrt{20}}$

$\left(\frac{1}{2}\right)^2 + \left(\frac{1}{4}\right)^2 = \frac{5}{16}$

$\sqrt{5/16}\ dz/dt = 1/4(-80) + 1/2(-90)$

$$\frac{dz}{dt} = \frac{-20 + -45}{\sqrt{5}/4} = -116$$

Lecture: Logarithmic Differentiation

Power Rule

Ex $y = x^3$

$y' = 3x^2$

VARIBLE IN EXPONENT

x

$y = 3^x$

$y' = 3^x \ln 3$

Logarithmic Differentiation

$$y = f(x)^{g(x)}$$

want y'

Ex $\quad y = x^{\cos x}$

$y = 3^x$

$y' = 3^x \ln 3$

$y = x^3$

$y' = 3x^2$

$$y = x^{\cos x}$$

log of both sides

$$\ln y = \ln x^{\cos x}$$

$$\ln y = \cos x \cdot \ln x$$

differentiate

$$\frac{d}{dx} \ln y = \frac{d}{dx}\left[\cos x \cdot \ln x\right]$$

product rule.

$$\frac{1}{y}\frac{dy}{dx} = \left[-\sin x \ln x + \frac{\cos x}{x}\right]$$

$$\frac{dy}{dx} = y\left[-\sin x \ln x + \frac{\cos x}{x}\right]$$

$$= x^{\cos x}\left[-\sin x \ln x + \frac{\cos x}{x}\right]$$

$$\ln(y) = \ln\left(\frac{x(x-1)}{x+2}\right)$$

$$\ln y = \ln x + \ln(x-1) - \ln(x+2)$$

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + \frac{1}{x-1} - \frac{1}{x+2}$$

$\ln y = \ln x^{\cos x}$ $\qquad y = x^{\cos x}$

① log of both sides $\qquad \frac{d\cancel{y}}{dx} =$

$\ln y = (\cos x)(\ln x)$

② Implicit differentiation

$\frac{1}{y} \cdot \frac{dy}{dx} = (-\sin x)(\ln x) + (\cos x)\left(\frac{1}{x}\right)$

$\frac{dy}{dx} = y\left[\frac{\cos x}{x} - \sin x \ln x\right]$

$= x^{\cos x}\left[\frac{\cos x}{x} - \sin x \ln x\right]$

$$\boxed{\frac{dy}{dx} = x^{\cos x}\left[\frac{\cos x}{x} - \sin x \ln x\right]}$$

$\ln y = \ln x^{\cos x}$

① log of both sides

$\ln y = \cos x \ln x$

② Implicit differentiation

$\frac{d}{dx}(\ln y) = \frac{d}{dx}(\cos x \cdot \ln x)$

$\frac{1}{y} \cdot \frac{dy}{dx} = \cos x \cdot \frac{1}{x} + \ln x \cdot \sin x$

$\frac{dy}{dx} = y\left(\frac{\cos x}{x} - \ln x \sin x\right)$

$\frac{dy}{dx} = x^{\cos x}\left(\frac{\cos x}{x} - \ln x \sin x\right)$

$$y = x^{\cos x}$$

$$y' = \cos x \cdot x^{\cos x - 1}$$

Ex $y = 3^x$

$\ln y = \ln 3^x$

$\frac{d}{dx}$ $\ln y = \frac{d}{dx} x \cdot \ln 3$

$\frac{1}{y} \frac{dy}{dx} = \ln 3$

$\frac{dy}{dx} = y \ln 3$

$\frac{dy}{dx} = 3^x \ln 3$

$y = B^x$

$y' = B^x \ln B$

$\ln y = \ln B^x$

$\ln y = x \ln B$

$\frac{1}{y} y' = \ln B$

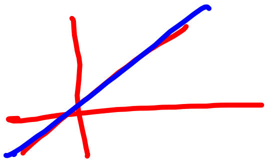
$y' = y \ln B$

$y' = B^x \ln B$

Project:

Review Analysis of Functions with Regressions

Linear

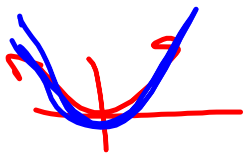


$y = ax + b$

$y' = a > 0$ increasing

$y'' = 0$ Not Concave

Quadratic



DEC INC

MIN

MAX

$y = ax^2 + bx + c$

$y' = 2ax + b > 0$ increasing

< 0 decreasing

$2ax + b = 0$

$x = -\frac{b}{2a}$ max/min

$y'' = 2a > 0$ Concave Up

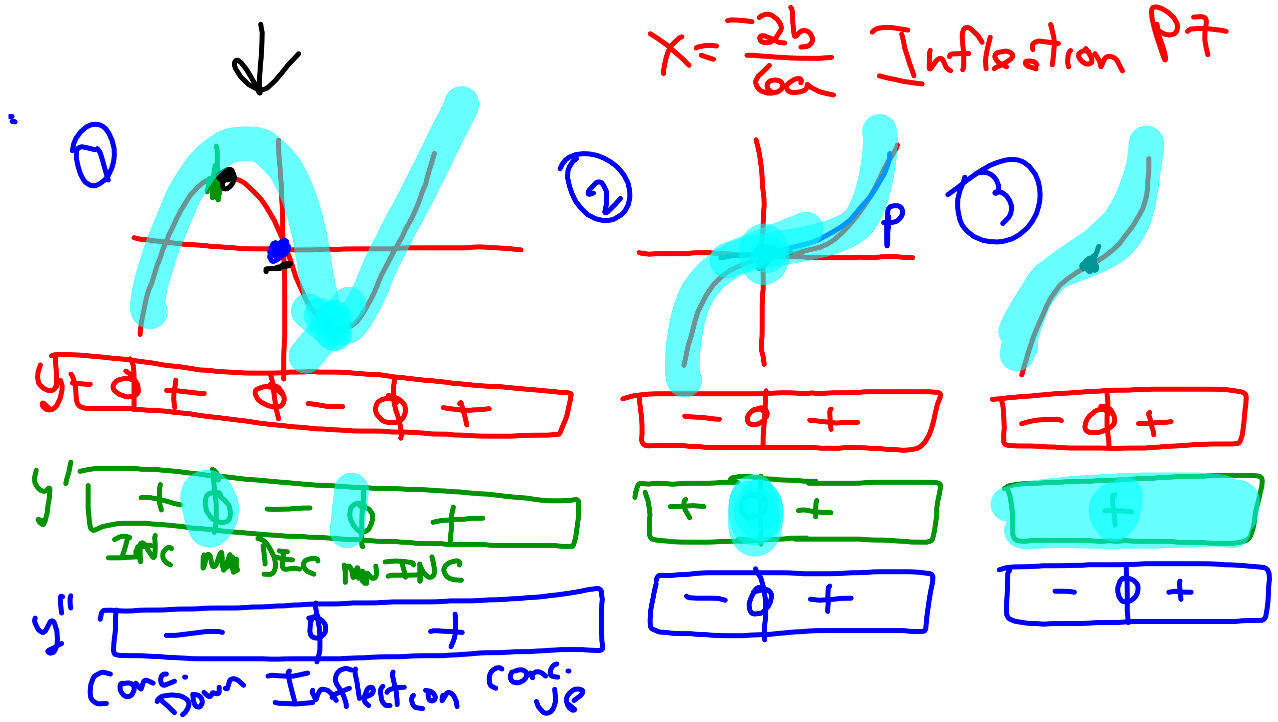
Cubic $y = ax^3 + bx^2 + cx + d$

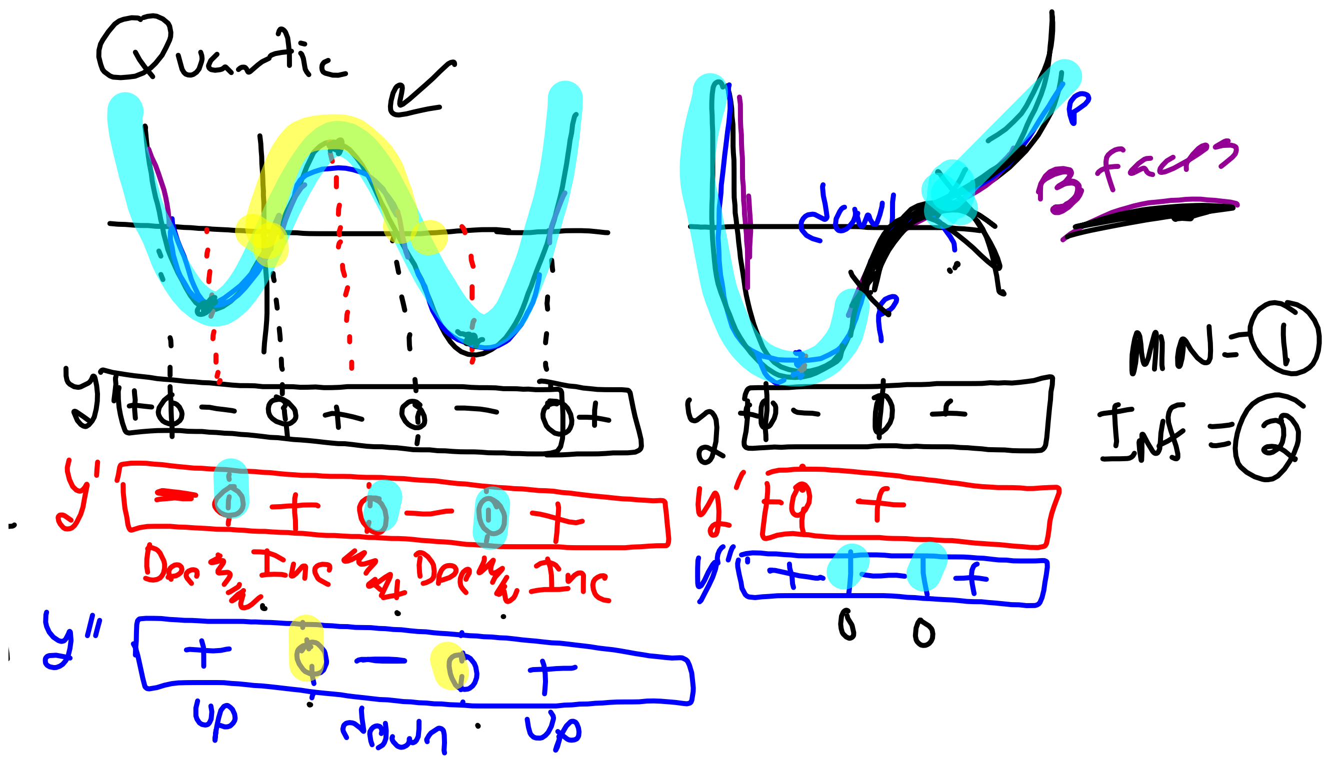
$y' = 3ax^2 + 2bx + c = 0$

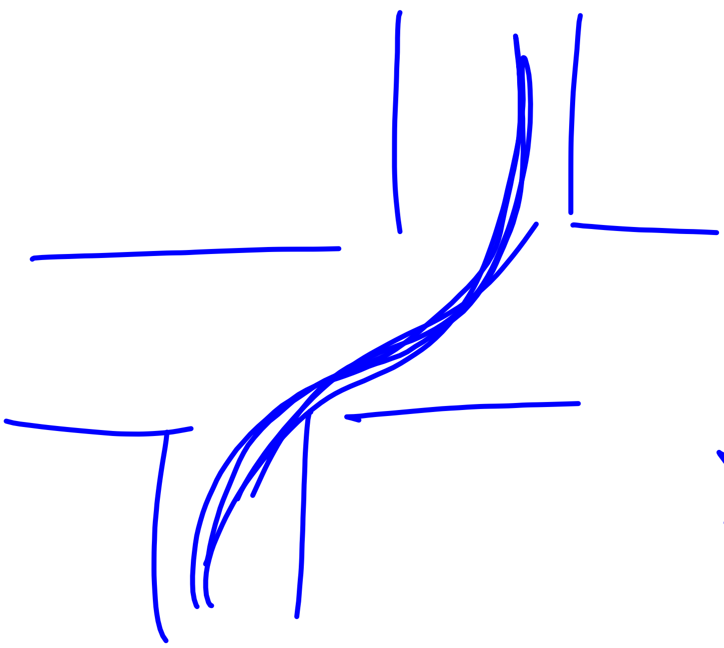
Two. Max/mins

$y'' = 6ax + 2b = 0$

$x = \frac{-2b}{6a}$ Inflection Pt



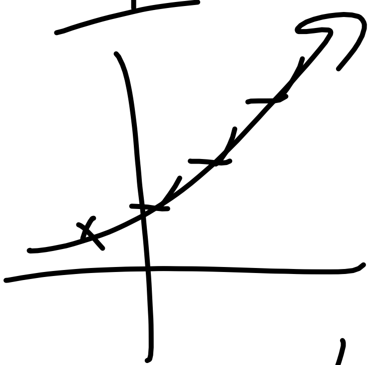

Quartic
y
+ 0 − 0 + 0 − 0 +
y'
− 0 + 0 − 0 +
Dec MIN Inc MAX Dec MIN Inc
y''
+ 0 − 0 +
up down up
down
P
P
3 facts
y
+ 0 − 0 +
y'
+ 0 +
y''
+ 0 − 0 +
0 0
MIN = 1
Inf = 2



J, g

$x^3 \quad x^5 \quad x^7$

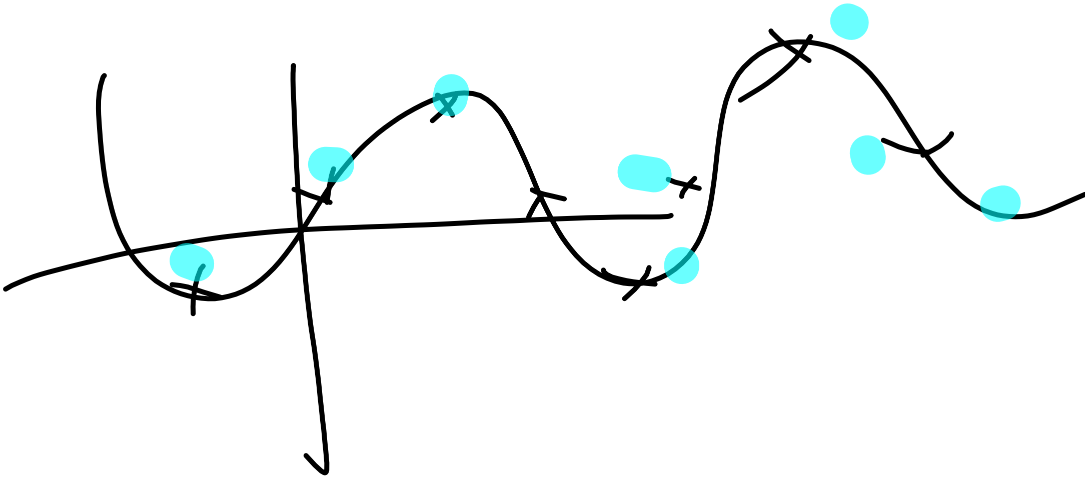
EXP



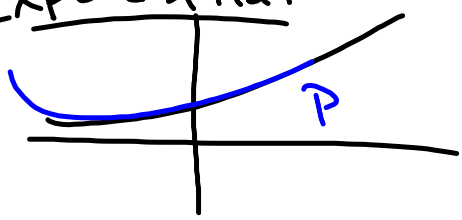
LOG

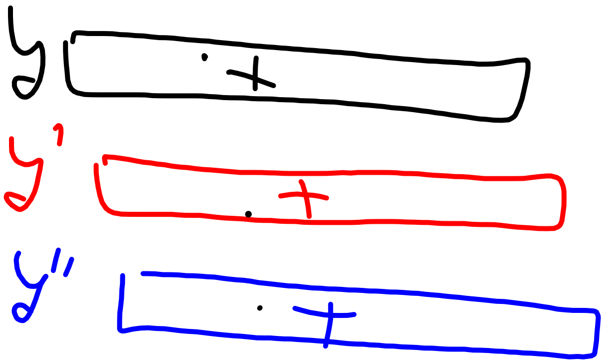
NO MAX/MIN/INT

Sin



Exponential

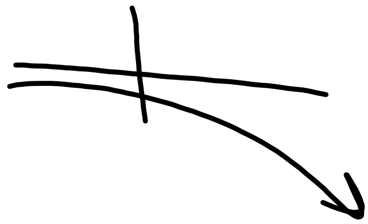




$$y = a \cdot b^x$$

$$y' = a b^x \ln b$$

$$y'' = a b^x (\ln b)^2$$

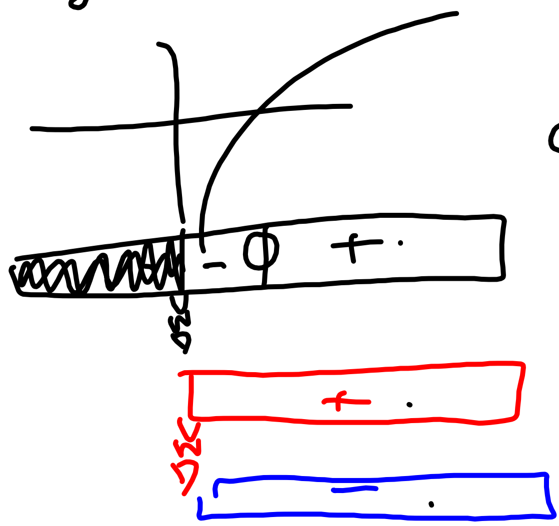


Natural Log.

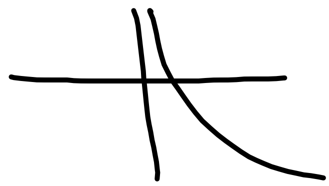
$$y = a + b \ln x$$

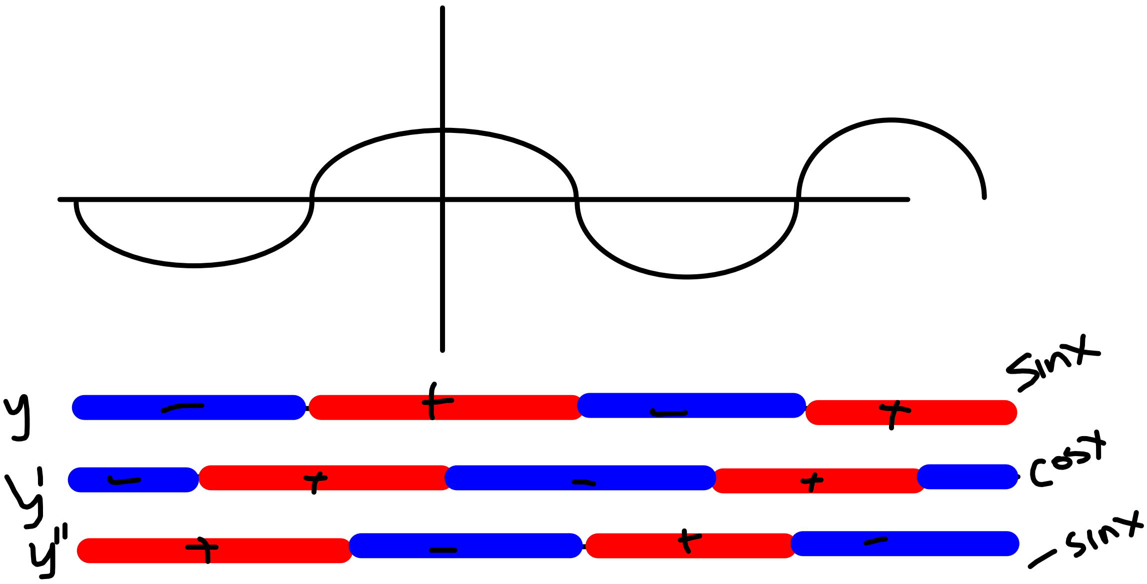
$$y' = b/x$$

$$y'' = -b/x^2$$



or




y
−
+
−
+
sinx
y'
−
+
−
+
cosx
y''
+
−
+
−
−sinx

Project:

y1= quartic regression   cubic

y2= derivative of y1 (by hand)   nderiv(y1,x,x)

y3= second derivative (by hand or nderiv)   nderiv(y2,x,x)

18. Using y'=0 to identify critical values a1,a2

X= 5

X - Y2/nderiv(Y2,X,X) → X

⋮

Using y''(a1) and y''(a2) to determine max/min

Critical Points	2.5	5.5	7.5
Y'' at critical Points	3.1 up	-1.7 down	1.5 up
Max or Min	MIN	MAX	MIN

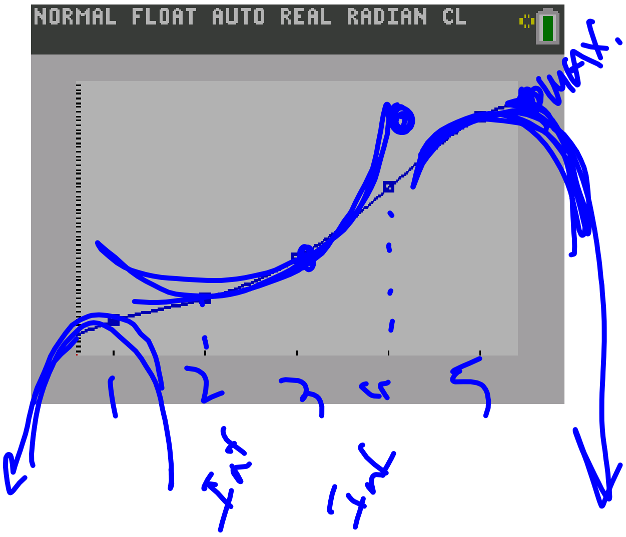
19. Using y''=0 to identify inflection points   Y''=0 at –b/(6a):________________

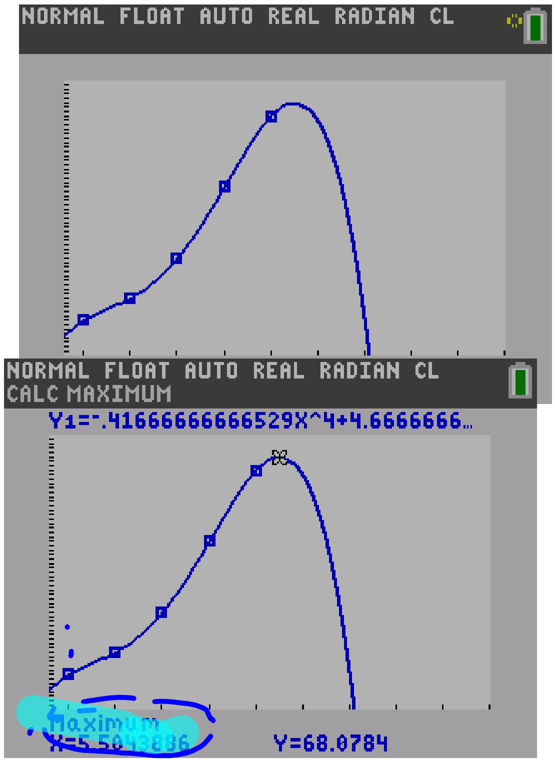
~~cubic~~

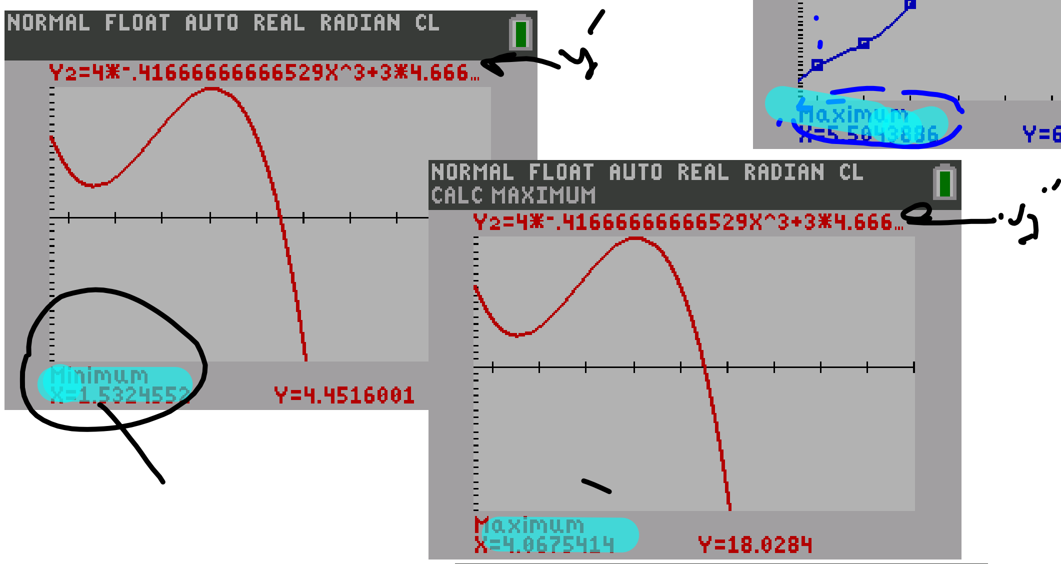
5 → X

X - Y3/nDeriv(Y3,X,X) → X

Inflection Points	4.5	6.5







NORMAL FLOAT AUTO REAL RADIAN CL

X	Y1	Y2	Y3
1	19	6.5	-8.167
1.53	21.712	4.4516	-.0312
2	24	5.6667	4.8333
3	33	12.833	7.8333
4.06	50.081	18.028	.09533
5	65	11.167	-16.17
5.5	68.078	.125	-28.42
6	64	-17.67	-43.17
7	19	-78.5	-80.17

X=

Inflection

Inflect

Max

$y'' = 0$

NORMAL FLOAT AUTO REAL RADIAN CL

X		Y1		Y2		Y3	
1		19	+	6.5	I	⁻8.167	Down
1.53	Inf	21.712	+	4.4516	I	⁻.0312	Down — almost 0.
2		24	+	5.6667	I	4.8333	Up
3		33	+	12.833	I	7.8333	Up
4.06	Inf	50.081	+	18.028	I	.09533	Up
5		65	+	11.167	I	⁻16.17	Down
5.5	MAX	68.078	+	1.25	crit.	⁻28.42	Down ·MAX
6		64	+	⁻17.67	D	⁻43.17	Down
7		19	+	⁻78.5	D	⁻80.17	Down

X=

$y'(5.5)=0$ $y''(5.5)<0$ MAX

Math.

$f'(5.6) = 0$

$f''(5.6) = -30$

Max 5.6

$f''(1.8) = 0$

$f''(4.1) = 0$

$\Big\}$ Inflect.

Words

According to Quartic regression, sales meet a maximum at $5.6.

with inflections at 1.8 and 4.1.