4.2 Constructing a second solution from a known solutions

Reduction of order

a2y''+a1y'+a0y=0

$u'' + q(x)u' = 0 \qquad w = u'$

$w' + q(x)w = 0$

Ex: $x^2y'' - 6y = 0$

$y = x^3$ is a solution

(verify: $x^2 \cdot 6x - 6 \cdot x^3 = 0$ yes)

Define $y = u(x)x^3$

$y' = u'x^3 + 3x^2u$

$y'' = u''x^3 + 3x^2u' + 3x^2u' + 6xu$

$u''x^3 + 6x^2u' + 6xu$

ORIGINAL

$x^2y'' - 6y = 0$

$x^2(u''x^3 + 6x^2u' + 6xu) - 6ux^3 = 0$

$u''x^5 + 6x^4u' + 6x^3u - 6ux^3 = 0$

$u''x^5 + 6x^4u' = 0$

$u'' + \frac{6}{x}u' = 0$

$w = u'$ $\quad w' = u''$

$w' + \frac{6}{x}w = 0$ (Reduced

Lin 1st O.

$e^{\int \frac{6}{x}dx} = e^{6\ln x} = x^6$

$x^6w' + 6x^5w = 0$

$x^6dw + 6x^5w\,dx$

$M_x = 6x^5 \quad N_w = 6x^5$ EQE

$d(x^6w) = 0$

$x^6w = c$

$x^6\frac{du}{dx} = c$

$du = cx^{-6}dx$

$u = \frac{cx^{-5}}{-5} + c_2$

$y = ux^3$

$\frac{y}{x^3} = c_1x^{-5} + c_2$

$y = c_1x^{-2} + c_2x^3$

$$\frac{a_2}{a_2}y'' + \frac{a_1}{a_2}y' + \frac{a_0}{a_2}y = 0$$

$$y'' + P(x)y' + Q(x)y = 0$$

$y = u(y_1)$ (known)

$y' = uy_1' + u'y_1$

$y'' = uy_1'' + u'y_1' + u'y_1' + u''y_1$

$y_1'' + Py_1' + Qy_1 = 0$

$uy_1'' + 2u'y_1' + u''y_1 + Puy_1' + Pu'y_1 + Quy_1 = 0$

$uy'' + Puy_1' + Quy_1 = 0$

$u(y_1'' + Py_1' + Qy_1) = 0$

$u''y_1 + 2u'y_1' + Pu'y_1 = 0$

$w = u' \quad w' = u''$

$w'y_1 + 2wy_1' + Pwy_1 = 0$

$y_1w' + (2y_1' + Py_1)w = 0$

$\frac{dw}{dx} \rightarrow w' + \left(\frac{2y_1'}{y_1} + P\right)w = 0$

$\frac{dw}{w} + \left(\frac{2y_1'}{y_1} + P\right)dx = 0$

$\ln|w| + 2\ln|y_1| + \int P\,dx = 0$

$\ln|wy_1^2| = -\int P\,dx + C$

$u' \leftarrow wy_1^2 = Ce^{-\int P\,dx}$

$u' = C\frac{e^{-\int P\,dx}}{y_1^2}$

$u = C_1\int \frac{e^{-\int P\,dx}}{y_1^2}\,dx + C_2$

$y = uy_1 = C_1y_1\int \frac{e^{-\int P\,dx}}{y_1^2}\,dx + C_2y_1$

$$y_2 = y_1\int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$$

$$y_2 = y_1\int \frac{e^{-\int P\,dx}}{y_1^2}\,dx$$

Ex $y_1 = x^2$ $\quad x^2y'' - 3xy' + 4y = 0$

$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = 0$ (P, Q)

$x^2\int \frac{e^{+\int \frac{3}{x}dx}}{x^4}\,dx$

$x^2\int \frac{e^{3\ln x}}{x^4}\,dx$

$x^2\int \frac{x^3}{x^4}\,dx = x^2\cdot\int \frac{1}{x}\,dx$

$x^2\ln x$ (Found)

$y_1 = x^2$ (Known) $\quad 1$

$$y = c_1x^2 + c_2x^2\ln x$$

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}dx \qquad (5$$

P = 0

Q = -1

as instructed, to find a second solution $y_2($.

$$y'' - 1y = 0; \quad y_1 = \cosh x$$

$$\int \frac{e^{\int 0\,dx}}{\cosh^2 x}dx$$

$y_2 =$ 3a [ ] sinh(x)

$\cosh x \int \cosh^{-2} x \, dx$

$\cosh x \int \operatorname{sech}^2 x \, dx$

$\cosh x \tanh x + C$

$\sinh x$

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}dx \qquad (5)$$

as instructed, to find a second solution $y_2(x)$.

$$4y'' - 12y' + 9y = 0; \quad y_1 = e^{3x/2}$$

$\frac{4}{4} \quad \frac{-12}{4} \quad \frac{9}{4}$

$$e^{3x/2}\int \frac{e^{+\int 12/4\,dx}}{e^{3x}}dx$$

$$e^{3x/2}\int 1\,dx = x e^{3x/2}$$

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)} dx \qquad (5$$

as instructed, to find a second solution y_2

$$y'' + 64y = 0; \quad y_1 = \cos 8x$$

$$\int \frac{e^0}{\cos^2 8x}\,dx = \int \sec^2(8x)\,dx$$

$$= \frac{\tan(8x)}{8} + c$$

$$y_2 = \cos 8x \cdot \frac{\tan(8x)}{8}$$

$$\sin(8x)$$

The indicated function $y_1(x)$ is a solution of the given differential

$$y_2 = y_1(x) \int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)} dx \qquad (5)$$

as instructed, to find a second solution $y_2(x)$.

$$(1 - 2x - x^2)y'' + 2(1 + x)y' - 2y = 0; \quad y_1 = x + 1$$

$y_2 =$ $x^2 + x + 2$

$$\int \frac{2(1+x)}{1-2x-x^2}\,dx \qquad u = 1-2x-x^2$$

$$du = (-2-2x)dx$$

$$-du = 2(1+x)dx$$

$$-\int \frac{1}{u}\,du = -\ln|1-2x-x^2|$$

$$e^{-\int P\,dx} = e^{\ln|1-2x-x^2|}$$

$$= 1-2x-x^2$$

$$\frac{e^{-\int P\,dx}}{y_1^2} = \int \frac{1-2x-x^2}{(x+1)^2}\,dx$$

Long Division

$$\frac{1-2x-x^2}{(x+1)^2} = -1 + \frac{2}{(x+1)^2}$$

$$-1 + \frac{2}{(x+1)^2}$$

$$x^2+2x+1 \;\overline{)\; -x^2-2x+1}$$

$$-x^2-2x-1$$

$$2$$

$$\int -1 + \frac{2}{(x+1)^2}\,dx$$

$$-x - 2(x+1)^{-1}$$

$$(x+1)\left(-x - 2\frac{1}{x+1}\right)$$

$$-x^2-x-2 \text{ or } x^2+x+2$$

$$y_2 = y_1(x)\int \frac{e^{-\int P(x)\,dx}}{y_1^2(x)}\,dx \qquad (5)$$

$P = \frac{2x}{1-x^2}$

as instructed, to find a second solution $y_2(x)$.

$(1 - x^2)y'' + 2xy' = 0; \quad y_1 = 1$

$y_2 =$ 8a $\boxed{\quad}$ $\boxed{x - \frac{x^3}{3}}$

$\int P\,dx$

$\int \frac{2x}{1-x^2}\,dx$

$-\ln|1-x^2| + C$

$e^{+\ln|1-x^2|} = 1-x^2$

$\int 1-x^2\,dx$

$1\left(x - \frac{x^3}{3}\right)$

Verify that $y_1(x) = x$ is a solution of $xy'' - xy' + y = 0$. Use re

- $y_2 = x \ln x + x^2 + \frac{x^3}{2\cdot 2!} + \frac{x^4}{3\cdot 3!} + \frac{x^5}{4\cdot 4!} + \cdots$
- $y_2 = -1 + x \ln x + \frac{x^2}{2} + \frac{x^3}{2\cdot 3!} + \frac{x^4}{3\cdot 4!} + \cdots$ 11a

$P = 1$

$e^{-\int P dx}$

e^{-x}

$x\int e^{-x}x^{-2}\,dx \qquad y_1\int \frac{e^{-\int P dx}}{y_1^2}\,dx$

$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!}$

$e^{-x} = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!}$

$x\int = \int x^{-2} - x^{-1} + \frac{1}{2} - \frac{x}{3!} + \cdots$

$x\left(\frac{x^{-1}}{-1} - \ln x + \frac{1}{2}x - \frac{x^2}{2\cdot 3!}\right)$

$-1 - x\ln x + \frac{x^2}{2} - \frac{x^3}{2\cdot 3!}$