Compute the exact work done by the force field $\mathbf{F}(x, y, z) = \langle 3y, 6xz, 4y \rangle$ acting on an object as it moves along the helix defined parametrically by $x = 3\cos t$, $y = 3\sin t$ and $z = 2t$, from the point $(3, 0, 0)$ to the point $(-3, 0, 2\pi)$.

$t=0 \qquad t=\pi$

$$W = \int F \cdot dr \rightarrow \int_{t=} F(t)\, r'(t)$$

$$r(t) = \langle 3\cos t, 3\sin t, 2t \rangle$$

$$dr = \langle -3\sin t, 3\cos t, 2 \rangle$$

$$F = \langle 9\sin t, 3(6t\cos t), 12\sin t \rangle \quad 3y \quad 6xz \quad 4y$$

$$\int_0^{\pi} -27\sin^2 t + 108t\cos^2 t + 24\sin t \, dt$$

$$= \text{Ans.}$$

Show Me

$$W = \int_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \int_C 3y\ dx + 6xz\ dy + 4y\ dz.$$

We have already provided parametric equations for the curve, but not the range of t-values. Notice that from $z = 2t$, you can determine that $(3, 0, 0)$ corresponds to $t = 0$ and $(-3, 0, 2\pi)$ corresponds to $t = \pi$. Substituting for x, y, z and $dx = -3\sin t\ dt$, $dy = 3\cos t\ dt$ and $dz = 2\ dt$, we have

$$W = \int_C 3y\ dx + 6xz\ dy + 4y\ dz$$

$$= \int_0^{\pi} [\underbrace{3(3\sin t)}_{3y}\underbrace{(-3\sin t)}_{x'(t)} + \underbrace{6(3\cos t)(2t)}_{6xz}\underbrace{(3\cos t)}_{y'(t)} + \underbrace{4(3\sin t)}_{4y}\underbrace{(2)}_{z'(t)}]\ dt$$

$$= \int_0^{\pi} (-27\sin^2 t + 108t\cos^2 t + 24\sin t)\ dt = 48 - \frac{27}{2}\pi + 27\pi^2,$$

where we used a computer algebra system to evaluate the final integral.

Evaluate the line integral.

$\int_C 7x\ ds$, where C is the line segment from $(1, 2)$ to $(7, 6)$

$\vec{ab} = \langle 6, 4 \rangle$

$$r(t) = \langle 1+6t, 2+4t \rangle$$

$$0 \le t \le 1$$

$$\int_{t=0}^{1} 7(1+6t)\sqrt{(x')^2+(y')^2}\, dt$$

$$\sqrt{6^2+4^2}$$

$$36+16=\sqrt{52}$$

$$\sqrt{52}\int_6^1 (7+42t)\, dt$$

Evaluate the line integral.

$\int_C 7x\ ds$, where C is the line segment from $(1, 2)$ to $(7, 6)$

$$x = 1 + 6t,\ y = 2 + 4t,\ 0 \le t \le 1$$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2}\ dt$$

$$= \sqrt{6^2 + 4^2}\ dt = \sqrt{52}\, dt$$

$$\int_C 7x\ ds = \int_0^1 7(1 + 6t)\sqrt{52}\ dt$$

$$= 28\sqrt{52}$$

Evaluate the line integral.

$\int_C 8z\ ds$, **where** C **is the line segment from (4, 0, 1) to (5, −8, 8)**

$x = 4 + t,\ y = -8t,\ z = 1 + 7t$

$0 \le t \le 1$

$$ds = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\ dt$$

$$= \sqrt{(1)^2 + (-8)^2 + (7)^2}\ dt = \sqrt{114}\ dt$$

$$\int_C 8z\ ds = \int_0^1 8(1 + 7t)\sqrt{114}\ dt = 36\sqrt{114}$$

Compute the work done by the force field F along the curve C.

$\mathbf{F}(x, y) = \langle 2x, 2y \rangle$, C **is the line segment from (2, 1) to (6, 4)**

$X = 2 + 4t$

$y = 1 + 3t$

$$\int F \cdot dr$$

$$\int_0^1 \langle 2(2+4t), 2(1+3t) \rangle \cdot \langle 4, 3 \rangle\, dt$$

$$\int_0^1 \left(8(2+4t) + 6(1+3t) \right) dt$$

Conservative

2D

$$\langle \overset{M}{P_x}, \overset{N}{P_y} \rangle \qquad \nabla P$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$P_{xy} = P_{yx}$$

3D

$$\nabla P$$

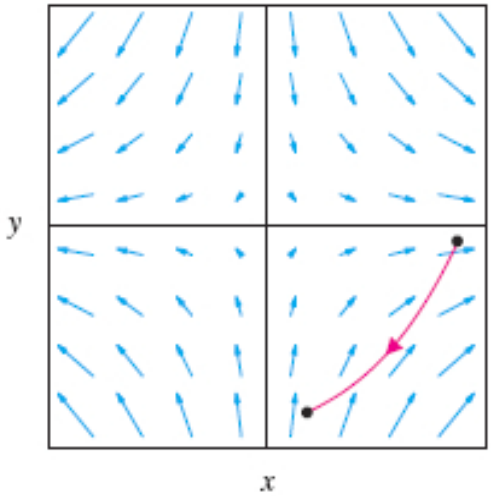
$$\left\langle \overset{M}{\frac{\partial P}{\partial x}}, \overset{N}{\frac{\partial P}{\partial y}}, \overset{P}{\frac{\partial P}{\partial z}} \right\rangle \qquad \langle \overset{M}{y}, \overset{N}{0}, \overset{P}{z} \rangle$$

$$\langle M_y - N_x, \; M_z - P_x, \; N_z - P_y \rangle$$

$$\langle 0, \; 0, \; 0 \rangle$$

$$\langle 1-0, \; 0-0, \; 0-1 \rangle$$

Using the graph, determine if the work done is positive, negative, or zero.



$\langle 0,1\rangle \cdot \langle 0,-1\rangle = -1$

Find the mass of a spring in the shape of the helix defined parametrically by $x = 8\cos t$, $y = 8t$, $z = 8\sin t$, for $0 \le t \le 10\pi$, with density $\rho(x, y, z) = 4y$.

- A. $12795\pi^2\sqrt{7}$
- B. $12802\pi^2\sqrt{3}$
- C. $12800\pi^2\sqrt{2}$
- D. $12797\pi^2\sqrt{2}$

$$M = \iiint_Q \rho(x,y,z)\,dV$$

$$\int_0^{10\pi} 4y \cdot ds$$

$$\sqrt{(x')^2+(y')^2+(z')^2}\,dt$$

$$\sqrt{8^2\sin^2 t + 8^2 + 8^2\cos^2 t}$$

$$\int_0^{10\pi} 4(8t) \cdot \sqrt{8^2+8^2}\,dt$$

$$8\sqrt{2}\,dt$$

$$256\sqrt{2}\int_0^{10\pi} t\,dt = 256\sqrt{2} \cdot \frac{t^2}{2}\Big|_0^{10\pi}$$

$$\frac{256\sqrt{2}}{2} \cdot (10\pi)^2$$

Show that for $F(x, y) = \langle \underbrace{2xy + 5}_{M}, \underbrace{x^2 + 6y + 3}_{N} \rangle$, the line integral $\int_C F(x, y) \cdot dr$ is independent of path. Then, evaluate the line integral for any curve C with initial point at $(-3, 4)$ and terminal point at $(2, 2)$.

Your Answer: [ ]

$M_y = 2x$
$N_x = 2x$ → conserv.

Conservative ⇒ Path Independent

Fundamental Theorem of Line Integrals.

$F = \nabla f$ $\qquad \int_C F \cdot dr = f(b) - f(a)$

$f_x = 2xy + 5$ $\qquad f_y = x^2 + 6y + 3$

$f = x^2y + 5x$ $\qquad f = x^2y + 3y^2 + 3y$

$f = x^2y + 5x + 3y^2 + 3y$

$f(2,2) - f(-3,4) = 8 + 10 + 12 + 6 = 36$
$-(36 - 15 + 48 + 12) = -81$

$36 - 81 =$
-45

Determine whether or not the line integral $\int_C (\underbrace{e^{3x} + 2x \sin y}_{M})\, dx + (\underbrace{6x^2 \cos y}_{N})\, dy$ is independent of path.

- A. The line integral is independent of path.
- B. The line integral is not independent of path.

$F \cdot dr$

$\langle M, N \rangle \cdot \langle dx, dy \rangle$

$\int_C M\,dx + N\,dy$

$M_y = N_x$

$2x\cos y \neq 12x\cos y$

Not conserv.

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$$\mathbf{F}(x, y, z) = \frac{\langle 1.74x, 4.09y, 2.04z \rangle}{\sqrt{1.74x^2 + 4.09y^2 + 2.04z^2}}$$

and C runs from (0, 5, 2) to (6, 4, 0).

$f = (1.74x^2 + 4.09y^2 + 2.04z^2)^{1/2}$

$f_x = \frac{1}{2}(1.7\ldots\ldots)^{-1/2} \cdot 2(1.74x)$

A potential function exists,
so the integral is independent of the path.
A potential function is

$$f(x, y, z) = \sqrt{1.74x^2 + 4.09y^2 + 2.04z^2}.$$

Thus

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \left[\sqrt{1.74x^2 + 4.09y^2 + 2.04z^2}\right]\Big|_{(6,4,0)}^{(0,5,2)}$$

$$\approx 0.809.$$

Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where

$\mathbf{F}(x, y) = \langle 2.77x^2y + 7, 2.77xy^2 \rangle$

and C is the bottom half–circle from (1, 0) to (−1, 0).

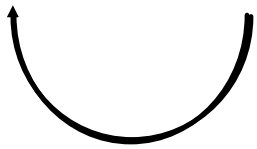
$M_y = 2.77x^2$ and $N_x = 2.77y^2$,

so the vector field is not conservative

and the line integral is not independent of the path.

For $0 \le t \le \pi$,

$$x = \cos t,$$
$$y = -\sin t.$$



$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_C (2.77x^2y + 7)\,dx + (2.77xy^2)\,dy$$

$$= \int_0^{\pi} \left[2.77(\cos^2 t)(-\sin t)(-\sin t) + 7(-\sin t) + 2.77(\cos t)(\sin^2 t)(-\cos t)\right] dt$$

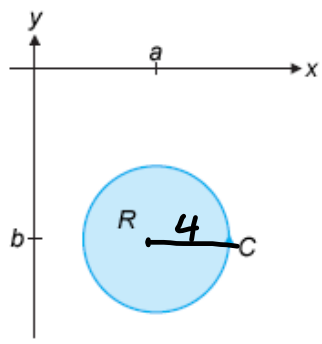
$$= 7\int_0^{\pi} -\sin t\,dt$$

$$= -14$$

Evaluate the line integral $\oint_C (6y - e^{\sin x})\,dx + [16x - \sin(y^3 + 5y)]\,dy$, where C is the circle of radius 4 centered at the point $(2, -8)$, as shown below.

$\oint_C$

closed curve.



$a = 2$ and $b = -8$

Your Answer:

$$\oint_C F \cdot dr = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dV$$

$$\iint_R (16-6)\,dV$$

$$10 \iint_R dV = 10 \cdot 16\pi$$

$$= 160\pi$$

$$\oint_C M\,dx + N\,dy$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dV$$

GREEN'S THEORY

Evaluate the indicated line integral, using Green's Theorem.

$\oint_C (x^2 - 8y)\,dx + y^2\,dy$, where C is the circle $x^2 + y^2 = 36$ oriented counterclockwise.

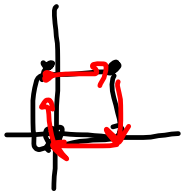
$$\frac{\partial N}{\partial x} = 0 \qquad \frac{\partial M}{\partial y} = -8$$

$$= 8\iint_R dV = 8 \cdot 36\pi$$

$$=$$

Evaluate the line integral $\oint_c x^2\,dx - x^3\,dy$ where C is the square contour from (0,0) to (0,1) to (1,1) to (1,0) to (0,0) using Green's Theorem.

Your Answer:



$$\frac{\partial N}{\partial x} = -3x^2 \qquad \frac{\partial M}{\partial y} = 0$$

$$\int_0^1\int_0^1 -3x^2\,dy\,dx = \int_0^1 dy \int_0^1 -3x^2\,dx$$

$$= -x^3\Big|_0^1 = -1$$

Change to 1

Use Green's theorem to evaluate the line integral

$$\oint_C (\tan x - 3y^3)\,dx + (3x^3 - \sin y)\,dy,$$

where C is the circle $x^2 + y^2 = 2$.

$$\frac{\partial N}{\partial x} = 9x^2 \qquad \frac{\partial M}{\partial y} = -9y^2$$

$$9\iint_R (x^2+y^2)\,dA \rightarrow 9\int_0^{2\pi}\int_0^{\sqrt{2}} r^2 \cdot r\,dr\,d\theta$$

Rdar

$$= 9 \cdot 2\pi \cdot \frac{r^4}{4}\Big|_0^{\sqrt{2}}$$

$$= 18\pi$$