Agenda

Review Quiz 13

Lecture : Surface Integrals

Lecture: Divergence Theorem

Stokes Theorem

Project Discussion

Evaluate the double integral $\iint_R (y - 9x)\, dA$ where R is the region in the uv-plane bounded by $x = \frac{-u+v}{12}$, $y = \frac{u+3v}{4}$, $5 \le u \le 6$, $4 \le v \le 5$.

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -1/12 & 1/12 \\ 1/4 & 3/4 \end{vmatrix}$$

$$y - 9x = -3/48 - 1/48$$

$$\frac{u+3v}{4} - \cancel{9}^{3}\left(\frac{-u+v}{\cancel{12}_{4}}\right) = 4/48 = 1/12$$

$$\frac{u+3v+3u-3v}{4} = \frac{4u}{4} = u$$

$$\int_{v=4}^{5}\int_{u=5}^{6} u \, \frac{1}{12}\, du\, dv$$

$$\frac{1}{12}\, \frac{u^2}{2}\Big|_5^6 \cdot (5-4)$$

$$\frac{36-25}{24} = \frac{11}{24}$$

$$\frac{\partial(x,\ y)}{\partial(u,\ v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{12} & \frac{1}{12} \\ \frac{1}{4} & \frac{3}{4} \end{vmatrix} = -\frac{1}{12}$$

$$\iint_R (y - 9x)\,dA = \int_4^5 \int_5^6 \left[\frac{1}{4}(u + 3v) - 9\left(\frac{1}{12}\right)(v - u)\right] \cdot \left|-\frac{1}{12}\right| du\ dv$$

$$= \frac{1}{12}\int_4^5 \int_5^6 u\ du\ dv$$

$$\frac{1}{12}\int_4^5 \frac{11}{2}\ dv = \frac{11}{24}$$

Conservative Vector Field

$\vec{F}\langle x, y\rangle$

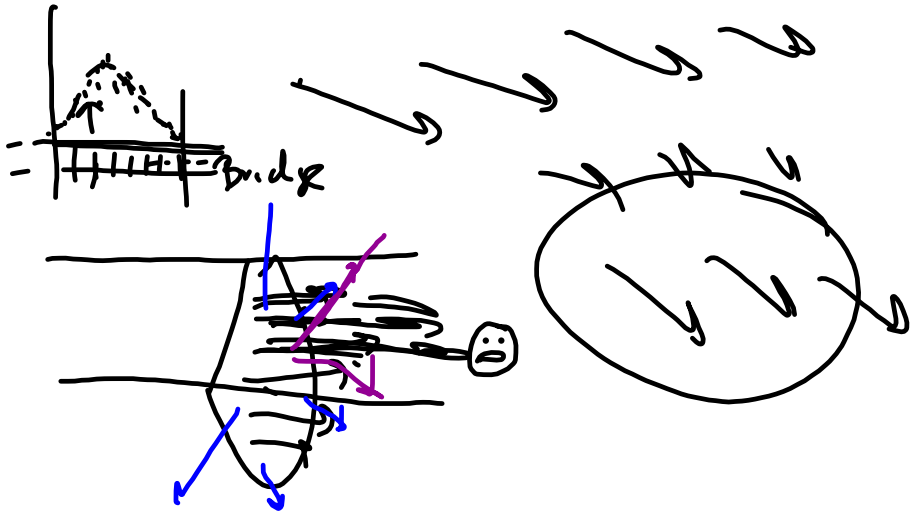
$$\int_C F \cdot dr = \phi(\text{final}) - \phi(\text{initial})$$

$$\nabla \phi = \langle M, N\rangle$$

$$= \langle M, N\rangle \qquad M_y = N_x$$

$$= \langle \phi_x, \phi_y\rangle = \nabla \phi$$

Surface Integral



plane

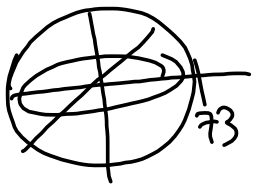
Find a parametric representation of the surface.
$z = 3x + 8y$

- A. $y = x$, ~~$x = y$~~, $z = 3x - 8y$, $-\infty < x < \infty$, $-\infty < y < \infty$
- ✓ B. $x = x$, $y = y$, $z = 3x + 8y$, $-\infty < x < \infty$, $-\infty < y < \infty$
- C. $y = 3x$, ~~$x = 8y$~~, $z = 3x + 8y$, $-\infty < x < \infty$, $-\infty < y < \infty$
- D. $x = 3x$, ~~$y = 8y$~~, $z = 3x - 8y$, $-\infty < x < \infty$, $-\infty < y < \infty$

$y = u \quad x = v \quad z = 3v + 8u$

Select the graph of the parametric surface.
$x = 73 \sin u \cos v$, $y = 73 \sin u \sin v$, $z = 73 \cos u$

$$x^2 = 73^2 \sin^2 u \cos^2 v$$
$$y^2 = 73^2 \sin^2 u \sin^2 v$$
$$x^2 + y^2 = 73^2 \sin^2 u \,(1)$$
$$z^2 = 73^2 \cos^2 u$$
$$x^2 + y^2 + z^2 = 73^2 (1)$$



sphere

Find the surface area of the given surface.

The portion of the cone $z = \sqrt{x^2 + y^2}$ below the plane $z = 3.6$.

$x = r\cos\theta \quad y = r\sin\theta \quad z = r$

There are other possible ways to solve this, but we can parametrize this surface by polar coordinates in the xy– plane:

$(x, y, z) = \mathbf{r} = (\mathbf{r} \cos \theta, \mathbf{r} \sin \theta, \mathbf{r})$

$0 \le \theta \le 2\pi, 0 \le \mathbf{r} \le 3.6$

$r = (x, y, z) \Rightarrow x(r, \theta)$

$$\mathbf{r}_\theta = \langle -\mathbf{r} \sin \theta, \mathbf{r} \cos \theta, 0 \rangle \leftarrow r_\theta$$
$$\mathbf{r}_\mathbf{r} = \langle \cos \theta, \sin \theta, 1 \rangle \leftarrow r_r$$
$$\mathbf{r}_\theta \times \mathbf{r}_\mathbf{r} = \langle \mathbf{r} \cos \theta, \mathbf{r} \sin \theta, -\mathbf{r} \rangle \leftarrow$$
$$\|\mathbf{r}_\theta \times \mathbf{r}_\mathbf{r}\| = \mathbf{r}\sqrt{2}$$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix}$$

$$\vec{i}(r\cos\theta) + \vec{j}(r\sin\theta) + \vec{k}(-r)$$
$$-r\sin^2\theta - r\cos^2\theta$$

$$\|r\cos\theta, r\sin\theta, -r\|$$
$$\sqrt{r^2\cos^2\theta + r^2\sin^2\theta + r^2}$$
$$\sqrt{2r^2} = r\sqrt{2}$$

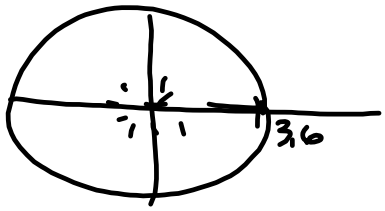
$$\text{Area} = \iint_R \|\mathbf{r}_\theta \times \mathbf{r}_\mathbf{r}\| \, dA_{\theta \mathbf{r}}$$

$\sqrt{2}$

$$= \iint_R \mathbf{r}\sqrt{2} \, dA_{\theta \mathbf{r}}$$

$$= \int_0^{2\pi} \int_0^{3.6} \mathbf{r}\sqrt{2} \, d\mathbf{r} \, d\theta$$

$$= \int_0^{2\pi} 6.48\sqrt{2} \, d\theta = 12.96\pi\sqrt{2}$$



Compute the surface area of the portion of the hyperboloid $x^2 + y^2 - z^2 = 64$ between $z = 3$ and $z = 7$. Round your answer to nearest tenth if needed.

$$x = 8\cos u \cosh v$$
$$y = 8\sin u \cosh v$$
$$z = 8\sinh v$$

$r = \langle x, y, z \rangle$

$$x^2 + y^2 = 8^2\cosh^2 v$$
$$z^2 = 8^2\sinh^2 v$$
$$x^2 + y^2 - z^2 = 8^2(\cosh^2 v - \sinh^2 v)$$
$$x^2 + y^2 - z^2 = 8^2$$

$x = 8\cos u \cosh v$

$y = 8\sin u \cosh v$

$z = 8\sinh v \qquad r = \langle x, y, z \rangle$

$$\vec{r}_u = \langle \underbrace{-8\sin u\cosh v}_{\hat{i}}, \underbrace{8\cos u\cosh v}_{\hat{j}}, \underbrace{0}_{\hat{k}} \rangle$$

$$r_v = \langle 8\cos u\sinh v, 8\sin u\sinh v, 8\cosh v \rangle$$

$$(64\cos u\cosh^2 v)\hat{i} + (64\sin u\cosh^2 v)\hat{j} +$$

$$-64\sin^2 u\cosh v\sinh v - 64\cos^2 u\cosh v\sinh v)\hat{k}$$

$$\|r_u \times r_v\| = \sqrt{64^2\cos^2 u\cos^4 v + 64^2\sin^2 u\cosh^4 v + 64^2\cosh^2 v\sinh^2 v}$$

$$= \sqrt{64^2\cosh^4 v + 64^2\cosh^2 v\sinh^2 v}$$

$$64\cosh v \cdot \sqrt{\cosh^2 v + \sinh^2 v}$$

$$\int_{u=0}^{2\pi}\int_{v=\sinh^{-1}(3/8)}^{\sinh^{-1}(7/8)} 64\cosh v\sqrt{\sinh^2 v + \cosh^2 v}\,dv\,du$$

$3 \le z \le 7$

$8\sinh v = 3 \qquad 8\sinh v = 7$

$\sinh v = 3/8 \qquad \sinh v = 7/8$

$v = \sinh^{-1}(3/8) \qquad v = \sinh^{-1}(7/8)$

$z = 8\sinh v$

$v = \sinh^{-1}\left(\frac{z}{8}\right)$

avoid question 1

1 out of 3 attempts

Set up the double integral and evaluate the surface integral $\iint_S g(x, y, z)\, dS$. $\sqrt{90}\, du\, dv$

$\iint_S xz\, dS$, S is the portion of the plane $z = 5x + 8y$ above the rectangle $1 \le x \le 5$, $1 \le y \le 8$

$1 \le v \le 8$ $\quad 1 \le u \le 5$

$x = u \quad y = v \quad z = 5u + 8v$

$r = \langle u, v, 5u + 8v \rangle$

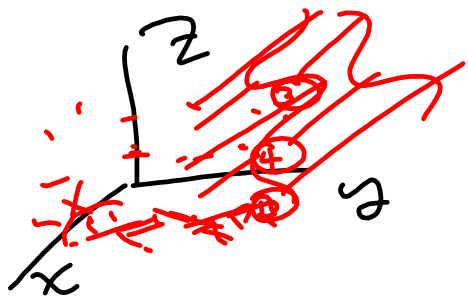
$r_u = \langle 1, 0, 5 \rangle \quad r_v = \langle 0, 1, 8 \rangle$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 5 \\ 0 & 1 & 8 \end{vmatrix} = \langle -5, -8, 1 \rangle$$

$$\|r_u \times r_v\| = \sqrt{25 + 64 + 1} = \sqrt{90}$$

$$\int_1^8 \int_1^5 \underbrace{u(5u + 8v)}_{\cancel{x}\ \cancel{z}} \sqrt{90}\, du\, dv$$

$g(x, y, z)$



$$\text{div } \vec{F} = \nabla \cdot \vec{F}$$

$$\vec{\nabla} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

Operator

$\vec{F} = \langle M, N, P \rangle$

$\nabla f = \langle f_x, f_y, f_z \rangle$

$$\frac{\partial}{\partial x} M + \frac{\partial}{\partial y} N + \frac{\partial}{\partial z} P = \text{div} \cdot F$$

Curl F

$$\vec{\nabla} \times \vec{F} \quad \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ M & N & P \end{vmatrix}$$

$$\langle P_y - N_z, M_z - P_x, N_x - M_y \rangle$$

(curl F = 0 F is conservative)

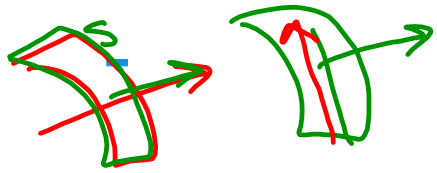
Divergence Theorem

$$\iint F \cdot n \, dS = \iiint_Q \text{div} F \, dV$$

Flux

Use the Divergence Theorem to compute

$$\iint_{\partial Q} \mathbf{F}\cdot\mathbf{n}\, dS \quad = \text{FLUX}$$



where Q is bounded by $z = x^2 + y^2$ and $z = 4$

and $\mathbf{F} = \langle 2x^3, 2y^3 - 3z, 3xy^2 \rangle$.

$$\langle 2x^3, 2y^3 - 3z, 3xy^2 \rangle$$

$$\frac{\partial}{\partial x} \quad \frac{\partial}{\partial y} \quad \frac{\partial}{\partial z}$$

$$6x^2 + 6y^2 + 0$$

Gym

$$\begin{aligned}
\nabla\cdot\mathbf{F} &= 6x^2 + 6y^2 + 0 \\
&= 6r^2 \\
\iint_{\partial Q} \mathbf{F}\cdot\mathbf{n}\, dS &= \iiint_Q \nabla\cdot\mathbf{F}\, dV \\
&= \iiint_Q 6r^2\, dV \\
&= \int_0^{2\pi}\int_0^2\int_{r^2}^4 6r^3\, dz\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^2 6r^3(4 - r^2)\, dr\, d\theta
\end{aligned}$$

div F

Jacob,

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{x^2+y^2}^{4} 6x^2 + 6y^2 \, dz\, dy\, dx$$

Use the Divergence Theorem to compute $\iint_{\partial Q} \mathbf{F}\cdot\mathbf{n}\, dS$, if Q is the cube $-4 \le x \le 4$, $-4 \le y \le 4$, $-4 \le z \le 4$ and $\mathbf{F} = \langle 3y^2, 7z - \cos x, z^3 - x \rangle$.

Your Answer: [ ]

$$\iiint \text{div } F\, dV$$

$$\text{div } F = 0 + 0 + 3z^2$$

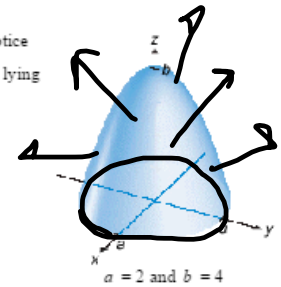
$$3\int_{x=-4}^{4}\int_{y=-4}^{4}\int_{z=-4}^{4} z^2\, dz\, dy\, dx$$

$$3(8)(8)\left.\frac{z^3}{3}\right|_{-4}^{4} = 64(64+64)$$

Show me

Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, **where** $\mathbf{F}(x, y, z) = \langle 4e^{z^2}, 3z - y, 6x \sin y \rangle$ **and where** S **is the portion of the paraboloid** $z = 4 - x^2 - y^2$ **above the** xy**-plane.**

In the figure to the right, we show the paraboloid S. Notice that the boundary curve is simply the circle $x^2 + y^2 = 4$ lying in the xy-plane. By Stokes' Theorem, we have



$a = 2$ and $b = 4$

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_{\partial S} \mathbf{F}(x, y, z) \cdot d\mathbf{r}$$

$$= \int_{\partial S} 4e^{z^2}\, dx + (3z - y)\, dy + 6x \sin y\, dz.$$

Now, we can parameterize ∂S by $x = 2\cos t$, $y = 2\sin t$, $z = 0$, $0 \le t \le 2\pi$. This says that on ∂S, we have $dx = -2\sin t$, $dy = 2\cos t$ and $dz = 0$. In view of this, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \int_{\partial S} 4(e^{z^2})\, dx + (3z - y)\, dy + 6x \sin y\, dz.$$

$$= \int_0^{2\pi} \{4e^0(-2\sin t) + [3(0) - 2\sin t](2\cos t)\}\, dt = 0,$$

where we leave the (straightforward) details of the calculation to you.

Use Stokes' Theorem to compute $\int\int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$.

S is the portion of the tetrahedron bounded by $x + y + 2z = 2$ and the coordinate planes with $z > 0$, $\mathbf{n}$ upward, $\mathbf{F} = \langle zy^4 - y^2, 3y - x^3, 6z^2 \rangle$.

∂S is the triangle with vertices at (0, 0, 0), (2, 0, 0), and (0, 2, 0) and is therefore made up of three line segments.

C_1: from (0, 0, 0) to (2, 0, 0).

$x = t, y = 0, z = 0, 0 \le t \le 2.$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{C_1} (zy^4 - y^2)\, dx + (3y - x^3)\, dy + 6z^2\, dz$$

$$= \int_0^2 [0 + 0 + 0]\, dt = 0$$

C_2: from (2, 0, 0) to (0, 2, 0).

References