Agenda

Review Quiz 13
Lecture: Surface Integrals
Lecture: Divergence Theorem
Stokes Theorem
Project Discussion

Evaluate the double integral \( \iint_{R} (y - x) \, dA \) where \( R \) is the region in the \( u-v \)-plane bounded by \( x = \frac{-u - v}{12}, \, y = \frac{u - 3v}{4}, \, -1 \leq u \leq 6, \, -1 \leq v \leq 5. \)

\[
\begin{align*}
\int_{0}^{5} \int_{2}^{5} \frac{2x}{5v} - \frac{2y}{5u} = & \left[ -\frac{1}{12} \frac{1}{12} \right] \left[ \frac{1}{14} \frac{1}{4} \right] \\
\int_{0}^{5} \frac{u + 3v}{4^3} - \frac{u + v}{5^2} = & \frac{1/48}{1/2} \\
\int_{0}^{5} \int_{2}^{5} \frac{u + 3v + 3u - 3v}{4} = & \frac{4u}{4} = u \\
\int_{2}^{5} \frac{u}{12} \, du \, dv = & \frac{u^2}{2} \bigg|_{2}^{5} \left(5 - 4 \right) \frac{36 - 25}{2^4} = \frac{11}{2^4}
\end{align*}
\]
\[ \frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{1}{12} & \frac{1}{12} \end{vmatrix} = -\frac{1}{12} \]

\[ \int \int_R (y - 9x) \, dA = \int_4^6 \int_5^6 \left[ \frac{1}{4} (u + 3v) - \Phi \left( \frac{1}{12} \right) (x - u) \right] \cdot \frac{1}{12} \, du \, dv \]

\[ = \int_4^6 \int_5^6 u \, du \, dv = \frac{11}{24} \]

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**Conservative Vector Field**

\[ \int_{\text{closed loop}} F \cdot dr = \Phi(\text{final}) - \Phi(\text{initial}) \]

\[ \nabla \Phi = \left\langle \begin{array}{c} m_x \\ n_y \end{array} \right\rangle \]

\[ = \left\langle \begin{array}{c} m_x \\ n_y \end{array} \right\rangle \quad m_y = n_x \]

\[ = \left\langle \Phi_x, \Phi_y \right\rangle = \nabla \Phi \]
Find a parametric representation of the surface.
\[ z = 3x + 8y \]

- A. \( y = x, \quad z = 3x - 8y, \quad -\infty < x < \infty, \quad -\infty < y < \infty \)
- B. \( x = x, \quad y = y, \quad z = 3x + 8y, \quad -\infty < x < \infty, \quad -\infty < y < \infty \)

\[ y = u, \quad x = v, \quad z = 3v + 8u \]
Select the graph of the parametric surface.
\[ x = 73 \sin u \cos v, \quad y = 73 \sin u \sin v, \quad z = 73 \cos u \]

\[ \begin{align*}
X^2 &= 73^2 \sin^2 u \cos^2 v \\
Y^2 &= 73^2 \sin^2 u \sin^2 v \\
X^2 + Y^2 &= 73^2 \sin^2 u \quad (1) \\
Z^2 &= 73^2 \cos^2 u \\
X^2 + Y^2 + Z^2 &= 73^2 \quad (1) \\
sphere
\end{align*} \]

Find the surface area of the given surface.

The portion of the cone \( z = \sqrt{x^2 + y^2} \) below the plane \( z = 3.6 \).

There are other possible ways to solve this, but we can parametrize this surface by polar coordinates in the \( xy \)-plane:

\( (x, y, z) = r = (r \cos \theta, r \sin \theta, r) \)

\( 0 \leq \theta \leq 2\pi, \quad 0 \leq r \leq 3.6 \)

\[ \begin{align*}
r_\theta &= \langle -r \sin \theta, r \cos \theta, 0 \rangle \\
r_r &= \langle \cos \theta, \sin \theta, 1 \rangle \\
r_\theta \times r_r &= \langle r \cos \theta, r \sin \theta, -r \rangle \\
\| r_\theta \times r_r \| &= r \sqrt{2}
\end{align*} \]
Compute the surface area of the portion of the hyperboloid \( x^2 + y^2 - z^2 - 64 \) between \( z = 3 \) and \( z = 7 \). Round your answer to nearest tenth if needed.

\[
\begin{align*}
X &= 8 \cos \mu \cosh \nu \\
y &= 8 \sin \mu \cosh \nu \\
z &= 8 \sinh \nu \\
r &= \langle X, y, z \rangle
\end{align*}
\]

\[
\begin{align*}
X^2 + y^2 &= 8^2 \cosh^2 \nu \\
z^2 &= 8^2 \sinh^2 \nu \\
x^2 + y^2 - z^2 &= 8^2 (\cosh^2 \nu - \sinh^2 \nu) \\
x^2 + y^2 - z^2 &= 8^2
\end{align*}
\]
\[ X = 8 \cos u \cosh v \\
Y = 8 \sin u \cosh v \\
Z = 8 \sinh v \quad r = \langle x, y, z \rangle \\
\overrightarrow{r} = \langle -8 \sin u \cosh v, 8 \cos u \cosh v, 0 \rangle \\
\overrightarrow{v} = \langle 8 \cos u \sinh v, 8 \sin u \sinh v, 8 \cosh v \rangle \\
(64 \cos u \cosh^2 v) \overrightarrow{r} + (64 \sin u \cosh^2 v) \overrightarrow{v} \\
- 64 \sin^2 u \cosh u \sinh v - 64 \cos^2 u \cosh u \sinh v \\
\| \overrightarrow{r} \times \overrightarrow{v} \| = \sqrt{64 \sin^2 u \cosh^4 v + 64 \sin^4 u \cosh^2 v + 64 \cos^2 u \cosh^2 v \sinh^2 v} \\
= \sqrt{64 \cosh^4 v + 64 \cos^2 u \cosh^2 v \sinh^2 v} \\
64 \cosh v \cdot \sqrt{\cos^2 u + \sinh^2 v} \\
\]

\[
\int_{\theta = 0}^{2\pi} \int_{\phi = 0}^{\sinh^{-1}(\frac{3}{8})} 64 \cosh v \sqrt{\sinh^2 v + \cosh^2 v} \, du \\
\]

\[ Z = 8 \sinh v \quad u = 0 \\
\theta = \sinh^{-1} \left( \frac{3}{8} \right) \\
3 \leq Z \leq 7 \\
8 \sinh v = 3 \quad 8 \sinh v = 7 \\
\sinh v = \frac{3}{8} \quad \sinh v = \frac{7}{8} \\
\theta = \sinh^{-1} \left( \frac{3}{8} \right) \quad \theta = \sinh^{-1} \left( \frac{7}{8} \right) \\
\]

avoid question 1
Set up the double integral and evaluate the surface integral \[ \iint_S g(x, y, z) \, dS \]

where \( S \) is the portion of the plane \( z = 5x + 8y \) above the rectangle \( 1 \leq x \leq 5 \), \( 1 \leq y \leq 8 \).

Let \( x = u \), \( y = v \), \( z = 5u + 8v \).

\[ \mathbf{r} = \langle u, v, 5u + 8v \rangle \]
\[ \mathbf{r}_u = \langle 1, 0, 5 \rangle \]
\[ \mathbf{r}_v = \langle 0, 1, 8 \rangle \]

The Jacobian is:
\[ \left| \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right| = 1 \]

\[ \left\| \mathbf{r}_u \times \mathbf{r}_v \right\| = \sqrt{25 + 64 + 1} = \sqrt{90} \]

\[ \iint_S \left( 5u + 8v \right) \sqrt{90} \, du \, dv \]

The function \( g(x, y, z) \) is shown on the diagram.
\[
\text{div } \vec{F} = \nabla \cdot \vec{F} \\
\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \\
\vec{F} = \left\langle M, N, P \right\rangle \\
\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial y} N - \frac{\partial}{\partial z} P, \frac{\partial}{\partial z} M - \frac{\partial}{\partial x} P, \frac{\partial}{\partial x} N - \frac{\partial}{\partial y} M \right\rangle \\
\text{Curl } \vec{F} = 0 \text{ if conservative}
\]

\textbf{Divergence Theorem}

\[
\oiint \text{Flux} = \iiint \text{div } \vec{F} \, dV
\]
Use the Divergence Theorem to compute
\[ \iiint_{Q} \text{div} \mathbf{F} \, dV \]
where \( Q \) is bounded by \( z = x^2 + y^2 \) and \( z = 4 \) and \( \mathbf{F} = (2x^3, 2y^3 - 3z, 3xy^2) \).

\[ \text{div} \mathbf{F} = \frac{\partial}{\partial x}(2x^3) + \frac{\partial}{\partial y}(2y^3 - 3z) + \frac{\partial}{\partial z}(3xy^2) \]
\[ = 6x^2 + 6y^2 - 3 \]
\[ = 6z^2 \]

\[ \iiint_{Q} \text{div} \mathbf{F} \, dV \]
\[ = \iiint_{Q} 6z^2 \, dV \]
\[ - \int_{0}^{2\pi} \int_{0}^{2} \int_{0}^{4} 6r^2 \, dz \, dr \, d\theta \]
\[ - \int_{0}^{2\pi} \int_{0}^{2} 6r^2 (4 - r^2) \, dr \, d\theta \]

Use the Divergence Theorem to compute \( \iiint_{Q} \text{div} \mathbf{F} \, dV \), if \( Q \) is the cube \(-1 \leq x \leq 4, -1 \leq y \leq 4, -1 \leq z \leq 4\) and \( \mathbf{F} = (3x^2, rz - \cos x, z^3 - x) \).

Your Answer:
Evaluate \( \iiint (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS \), where \( \mathbf{F}(x, y, z) = \langle 4e^z, 3z - y, 6x \sin y \rangle \) and where \( S \) is the portion of the paraboloid \( z = 4 - x^2 - y^2 \) above the xy-plane.

In the figure to the right, we show the paraboloid \( S \). Notice that the boundary curve is simply the circle \( x^2 + y^2 = 4 \) lying in the xy-plane. By Stokes' Theorem, we have

\[
\oint_{\partial S} \mathbf{F} \cdot d\mathbf{r} = \int_{S} (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS
\]

where \( \partial S \) is the boundary curve of \( S \).

Now we can parameterize \( \partial S \) by \( x = 2 \cos t, \ y = 2 \sin t, \ z = 0, \ 0 \leq t \leq 2\pi \). This says that on \( \partial S \), we have \( dx = -2 \sin t \, dt, \ dy = 2 \cos t \, dt \) and \( dz = 0 \). In view of this, we have

\[
\int_{\partial S} (\nabla \times \mathbf{F}) \cdot d\mathbf{r} = \int_{0}^{2\pi} \left[ 4e^{2 \cos t} \sin t + (3z - y) \cos t + 6x \sin y \right] dt = 0
\]

where we leave the (straightforward) details of the calculation to you.
Use Stokes' Theorem to compute $\int \int_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} \, dS$.

$S$ is the portion of the tetrahedron bounded by $x + y + 2z = 2$ and the coordinate planes with $z > 0$, $n$ upward, $\mathbf{F} = \begin{pmatrix} x^3 y - y^2 z \cr 3y - x^3 \cr 6z \end{pmatrix}$.

$\partial S$ is the triangle with vertices at $(0, 0, 0)$, $(2, 0, 0)$, and $(0, 2, 0)$ and is therefore made up of three line segments.

$C_1$: from $(0, 0, 0)$ to $(2, 0, 0)$.

$x = t, y = 0, z = 0, 0 \leq t \leq 2$.

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} (xy^2 - y^2 z) \, ds + (3y - x^3) \, dy + 6z^2 \, dz$$

$$= -\int_0^2 [0 + 0 + 0] \, dt = 0$$

$C_2$: from $(2, 0, 0)$ to $(0, 2, 0)$. 

$C_3$: from $(0, 0, 0)$ to $(0, 2, 0)$. 

$C_4$: from $(0, 2, 0)$ to $(0, 0, 0)$.