Agenda

Review Quiz 12

Lecture Conservative and path

Independance

Lecture Greens Theorem

Review Quiz 12

Compute the exact work done by the force field \( \mathbf{F}(x, y, z) = (y, x, z) \) acting on an object as it moves along the path defined parametrically by \( x = \cos t, \)
\( y = 3 \sin t \) and \( z = 2t, \) from the point \( (3, 0, 0) \) to the point \( (-3, 3, 2). \)

\[
W = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{\pi} \mathbf{F}(t) \cdot \mathbf{r}'(t)
\]

\[
\mathbf{r}(t) = \langle 3 \cos t, 3 \sin t, 2t \rangle
\]

\[
d\mathbf{r} = \langle -3 \sin t, 3 \cos t, 2 \rangle
\]

\[
\mathbf{F} = \langle 3 \sin t, 3 \cos t, 2 \rangle
\]

\[
\int_{0}^{\pi} -27 \sin^2 t \, dt + 108 \pi t + 24 \sin 2t \, dt
\]

\[
= 
\]

We have already provided parametric equations for the path, but not the range of \( t \)-values.

To find the range of \( t \), we can determine that \( (3, 0, 0) \) corresponds to \( t = 0 \) and \( (-3, 3, 2) \) corresponds to \( t = \pi/2 \). Substituting for \( x, y, \) and \( z \) and \( dt = \frac{1}{2} \, dt \),
\( dt \rightarrow \frac{1}{2} \, dt \) and \( dt = \frac{1}{2} \, dt \) we have

\[
\int_{0}^{\pi} -27 \sin^2 t \, dt + 108 \pi t + 24 \sin 2t \, dt
\]

where we used a computer algebra system to evaluate the final integral.
Evaluate the line integral.
\[ \int_C \mathbf{F} \cdot d\mathbf{r}, \text{ where } C \text{ is the line segment from } (1, 2) \text{ to } (7, 0) \]
\[ \mathbf{F} = \langle 4y \rangle \]
\[ \mathbf{r}(t) = \langle 1 + 6t, 2 + 4t \rangle \]
\[ 0 \leq t \leq 1 \]
\[ \int_0^1 7(1+6t) \sqrt{(1+6t)^2+4^2} \, dt \]
\[ \frac{7}{6^2+4^2} \cdot 7 + 42t \, dt \]

Evaluate the line integral.
\[ \int_C \mathbf{F} \cdot d\mathbf{r}, \text{ where } C \text{ is the line segment from } (1, 2) \text{ to } (7, 0) \]
\[ x = 1 + 6t, \quad y = 2 + 4t, \quad 0 \leq t \leq 1 \]
\[ ds = \sqrt{(1+6t)^2+4^2} \, dt \]
\[ = \sqrt{114} \, dt \]
\[ \int_0^1 7(1+6t) \sqrt{114} \, dt \]
\[ = 28 \sqrt{114} \]

Evaluate the line integral.
\[ \int_C 8z \, ds, \text{ where } C \text{ is the line segment from } (4, 0, 1) \text{ to } (5, -8, 8) \]
\[ x = 4 + t, \quad y = -8t, \quad z = 1 + 7t \]
\[ 0 \leq t \leq 1 \]
\[ ds = \sqrt{(1)^2 + (-8)^2 + (7)^2} \, dt \]
\[ = \sqrt{114} \, dt \]
\[ \int_0^1 8(1+7t) \sqrt{114} \, dt \]
\[ = 36 \sqrt{114} \]
Compute the work done by the force field $F$ along the curve $C$. $F(x, y) = \langle 2x, 2y \rangle$, $C$ is the line segment from $(2, 1)$ to $(6, 4)$

\[
\begin{align*}
&\mathbf{X} = 2 + 4t \\
&\mathbf{Y} = 1 + 3t \\
\int_0^1 &\mathbf{F} \cdot \mathbf{v} \\
&\int_0^1 \langle 2(2+4t), 2(1+3t) \rangle \cdot \langle 4, 3 \rangle \, dt \\
&\int_0^1 (8(2+4t) + 6(1+3t)) \, dt \\
&\end{align*}
\]

Conservative

\[
\begin{align*}
\langle M, N \rangle &= \langle P_x, P_y \rangle \\
\frac{\partial M}{\partial y} &= \frac{\partial N}{\partial x} \\
P_{xy} &= P_{yx}
\end{align*}
\]
Show that for \( F(x, y) = \langle 2xy - 5, x^2 + 6y - 3 \rangle \), the line integral \( \int_C F(x, y) \cdot dr \) is independent of path. Then, evaluate the line integral for any curve \( C \) with initial point at \((-3, 4)\) and terminal point at \((2, 2)\).

Your Answer:

\[
\begin{align*}
M_y &= 2x \\
N_x &= 2x \\
\text{Conservative} &\Rightarrow \text{Path Independent}
\end{align*}
\]

\[
\begin{align*}
f_x &= 2xy - 5 \\
f_y &= x^2 + 6y + 3 \\
f &= x^2y - 5x + (56) \\
f &= x^2y + 3y^2 + 3y \\
\nabla f &= \langle 2xy - 5, x^2 + 6y + 3 \rangle \checkmark
\end{align*}
\]
\[ \langle 2, 2 \rangle \]

\[
\begin{align*}
\int_0^1 \langle 2(-3+5t)(4-2t)+5,(3+5t)^2+6(4-2t)+3 \rangle dt \\
\int_0^1 \langle 10(-3+5t)(4-2t)+25-2(3+5t)^2-12(4-2t) \rangle dt \\
= -45
\end{align*}
\]

\[
f(x, y) = 2xy + 5x^2 + 3y^2 + 3y
\]

\[
f(2, 2) - f(-3, 4) = 8 + 10 + 12 + 6 = 36
\]

\[
-(36 - 15 + 48 + 12) = -91
\]

\[
f' = 2xy + 5x + 3y^2 + 3y
\]

\[
36 - 91 = -55
\]

\underline{Fundamental Theorem of Line Integrals}:

\[
\int_C \vec{F} \cdot d\vec{r} = 0 \text{ if } \vec{F} \text{ is conservative.}
\]

Then \[
\int_C \vec{F} \cdot d\vec{r} = f(b) - f(a)
\]
Determine whether or not the line integral \( I_C \left( e^{3x} + 2x \sin y \right) dx + (6x^2 \cos y) \, dy \) is independent of path.

- A. The line integral is independent of path.
- B. The line integral is not independent of path.

\[
\nabla \cdot \mathbf{F} = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 0
\]

\[
\mathbf{F} = \left\langle m, n \right\rangle
\]

\[
M = \frac{\partial}{\partial x} (12x \cos y) = 12x \cos y
\]

\[
N = \frac{\partial}{\partial y} (12x \cos y) = -12x \sin y
\]

\[
\mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot d\mathbf{r}
\]

Evaluate \( \mathbf{F} \cdot d\mathbf{r} \), where

\[
\mathbf{F}(x, y, z) = \left( \frac{1.74x, 4.96y, 1.04z}{\sqrt{1.74x^2 + 4.09y^2 + 2.04z^2}} \right)
\]

and \( C \) runs from \((0, 5, 2)\) to \((6, 4, 0)\).

A potential function exists, so the integral is independent of the path. A potential function is

\[
f(x, y, z) = \sqrt{1.74x^2 + 4.09y^2 + 2.04z^2}
\]

Thus

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \left[ \sqrt{1.74x^2 + 4.09y^2 + 2.04z^2} \right]_{(0, 5, 2)}^{(6, 4, 0)} \approx 0.809
\]

\[
\begin{align*}
\mathbf{F}(6, 4, 0) &= f(6, 4, 0) \\
\mathbf{F}(0, 5, 2) &= f(0, 5, 2)
\end{align*}
\]

\[
\text{END}
\]

Evaluate \( \mathbf{F} \cdot d\mathbf{r} \), where

\[
\mathbf{F}(x, y, z) = \left( \frac{1.74x, 4.96y, 1.04z}{\sqrt{1.74x^2 + 4.09y^2 + 2.04z^2}} \right)
\]

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\]

\[
\begin{align*}
\mathbf{F}(6, 4, 0) &= f(6, 4, 0) \\
\mathbf{F}(0, 5, 2) &= f(0, 5, 2)
\end{align*}
\]
Example

Evaluate \[ \int_C \mathbf{F} \cdot d\mathbf{r} \], where \[ \mathbf{F}(x, y) = (2.77x^2 y + 7, 2.77xy^2) \]

and \( C \) is the bottom half-circle from \((1, 0)\) to \((-1, 0)\).

\( M_y = 2.77x^2 \) and \( N_x = 2.77y^2 \), so the vector field is not conservative and the line integral is not independent of the path.

For \( 0 \leq t \leq \pi \),
\[
\begin{align*}
x &= \cos t, \\
y &= -\sin t.
\end{align*}
\]

\[
\int_C \mathbf{F} \cdot d\mathbf{r}
\]

\[
= \int_C (2.77x^2 y + 7) \, dx + (2.77xy^2) \, dy
\]

\[
= \int_0^\pi \left[ 2.77(\cos^2 t)(-\sin t)(-\sin t) + 7(-\sin t) + 2.77(\cos t)(\sin^2 t)(-\cos t) \right] \, dt
\]

\[
= 7\int_0^\pi -\sin t \, dt
\]

\[
= -14
\]
Green's Theorem

Notation

∫
C
F \cdot dr = \iint_{R} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \, dx \, dy

Closed Curve
Starts at ends in same place.
Assume not conservative.
Evaluate the line integral \( \int_C (6y - 2 \sin x) \, dx + [16x - \sin (y^3 - 5y)] \, dy \), where \( C \) is the circle of radius 4 centered at the point \((2, -8)\), as shown below.

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} = 10 \oint_C (16 - 6) \, dx + 10 \oint_C (6 \cdot 16) \, dy = 160 \pi \]

**Closing Curve**

**Area of Region**

\[ \pi r^2 = \pi \]
Example
Evaluate the indicated line integral, using Green’s Theorem.
\[ \oint_C (x^2 - 8y) \, dx + y^2 \, dy \], where \( C \) is the circle \( x^2 + y^2 = 36 \) oriented counterclockwise.

\[ \frac{2N}{2y} = 0 \quad \frac{2M}{2y} = -8 \quad \frac{2N}{2x} - \frac{2m}{2y} = 8 \]

\[ = 8 \int \int_R N \, dx \, dy = 8 \cdot 3 \cdot 11 \]

Example
Evaluate the line integral \( \int_C x^2 \, dx - x^3 \, dy \) where \( C \) is the square contour from \((0,0)\) to \((0,1)\) to \((1,1)\) to \((1,0)\) to \((0,0)\) using Green’s Theorem.

Your Answer: __________

\[ \frac{2N}{2y} = -3x^2 \quad \frac{2M}{2y} = 0 \]

\[ \int \int_R -3x^2 \, dy \, dx = \int_0^1 \int_0^1 -3x^2 \, dx \, dy \]

\[ = -x^3 \bigg|_0^1 = -1 \]

\[ \text{Left Hand Rule.} \]
Use Green's theorem to evaluate the line integral 
\[ \int_C (\tan x - 3y^3) \, dx + (3x^3 - \sin y) \, dy, \]
where \( C \) is the circle \( x^2 + y^2 = 2 \).

\[
\begin{align*}
\frac{2N}{2x} &= 9x^2 \\
\frac{2m}{2y} &= -9y^2 \\
9 \int \int (x^2 + y^2) \, dA &= 9 \int_0^{\pi/2} \int_0^2 r^2 \, r \, dr \, d\theta \\
P_{\text{div}} &= 9 \cdot 2\pi \cdot \frac{5^2}{4} \\
&= 18\pi
\end{align*}
\]

Project

\[
\begin{align*}
\mathbf{r} &= \langle 75 \cos \theta, 25 \sin \theta, 75 \sin \theta \rangle \\
\mathbf{F} &= \langle 0, 0, -9.8 \text{ m/s}^2 \rangle \\
\mathbf{F} \cdot d\mathbf{r} &= \int 0 \, d\theta \\
\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{\pi/2} (-9.8)(75 \cos \theta) \, d\theta \\
\text{Is } \mathbf{F} \text{ conservative?}
\end{align*}
\]