Agenda
Overview
Review Quiz 6
Lecture Optimization
- Boundary Value Problems
Lecture Error
Project: 3D printer?
Introduce Lagrange Multipiers

Calc 3 Overview-Multidimensional
Vectors (magnitude/direction=slope), Parametric lines, surfaces
Partial Derivatives
- Tangent lines/planes, "Linearization", Chain Rule, Implicit, Max/Min, Boundaries, Constraints
Integrals (later)
Quiz 6 Review

Please fully reduce the final answer.

Compute the linear approximation of the function \( f(w, x, y, z) = 3w^2xy - e^{6wy}z \) at the point \((-9, 4, 3, 0)\).

\[
L(-9, 4, 3, 0) = \left[ \begin{array}{c}
-648 \\
729 \\
972 \\
162 \\
\end{array} \right] =
\begin{array}{c}
25 = 6wxy - e^{6wy}z \\
9w = 3w^2y = 3(9)(3) \\
9y = 3w^2x - e^{6wy}z = 3(9)(4) \\
2z = -e^{6wy}z = -e^0 = 1 \\
\end{array}
\]

\[
L = -648(w + 9) + 729(x - 4) + 972(y - 3) + 162z
\]
Use implicit differentiation on the equation

\[ 6e^{x^2y^2} - 4x^2 + x \cos y = 3 \]

to find \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

\[ F(x, y, z) = 6e^{x^2y^2} - 4x^2 + x \cos y - 3 \]

\[ \frac{2z}{2x} = -\frac{F_x}{F_z} \]

\[ \frac{2z}{2y} = -\frac{F_y}{F_z} \]

\[ 0 = \frac{2F_x}{2x} + \frac{2F_y}{2y} + \frac{2F_z}{2z} \]

\[ 0 = F_x - F_z \frac{2z}{2x} \]

Find equations of the tangent plane and normal line to the surface at the given point.

\( z^2 - x^2 + y^2 \) at \((3, 5, -5)\)

\[ f(x, y, z) = x^2 + y^2 - z^2 \]

\[ \nabla f = \langle 2x, 2y, -2z \rangle \]

\[ \nabla f(3, 5, -5) = \langle 6, 10, 10 \rangle \]

\( 6(x - 3) + 10(y - 5) + 10(z + 5) = 0 \)

The normal line has parametric equations

\[ x = 3 + 6t \]

\[ y = 5 + 10t \]

\[ z = -5 + 10t \]
Sec. Ex. 31 - 12.4 Section Exercise 31

Your response: 

Find the total differential of \( f(x, y) = ye^{6x} + \sin 2x \).

- A. \( dz = (6y e^{6x} + 2 \cos 2x)dx + (e^{6x})dy \)
- B. \( dz = (y e^{6x} + 2 \cos 2x)dx + (e^{6x})dy \)
- C. \( dz = (6y e^{6x} + 2 \cos x)dx + (e^{6x})dy \)
- D. \( dz = (6y e^{6x} + 2 \cos 2x)dx + (e^{6x})dy \)

\( \frac{\partial z}{\partial x} \quad \frac{\partial z}{\partial y} \)

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Connect?

Sec. Ex. 3 - Vector Calculus

Your response: 

Locate all critical points and classify them.

\( f(x, y) = 12x^3 - 36xy + 12y^3 \)

- A. (0, 0) is a saddle point and (1, 1) is a local minimum.
- B. (0, 0) is a local maximum and (-1, -1) is a saddle point.
- C. (0, 0) is a saddle point and (-1, -1) is a local minimum.
- D. (0, 0) is a local minimum and (1, 1) is a local maximum.

\( f_{xy} = 72y \quad f_{xx} = 72x > 0 \) (concave up)

\( f_{yy} = -36 \quad D = (72x)(72y) - 36^2 \)

\( D(11) = 72^3 - 36^3 > 0 \)
Optimization

Boundary Problems

1. Find \( \nabla f = \mathbf{0} \)
   \[ \langle f_x, f_y \rangle = \langle 0, 0 \rangle \]

2. Each Boundary
   \( x = 0 \) or \( y = 3 \)
   \( y = x \)
   Use regular \( f'(x) = 0 \)

3. Find the End Points (or corners)

4. Plug them all in
   Find big & small.
What if boundary is a circle of radius 3, centered at origin?

2. Max/Min on Boundary??

\[ x = 3 \cos t \quad 0 \leq t \leq 2\pi \]
\[ y = 3 \sin t \]

If maybe

\[ x = \cos t \quad 0 \leq t < \pi \]
\[ y = \sin t \quad -1 \leq x \leq 1 \]
\[ y = 0 \]

Construct the function \( d(x, y) \) giving the distance from a point \((x, y, 1)\) on the paraboloid \( z = x^2 + y^2 \) to the point \((1, 1, 1)\). Then determine the points that minimize \( d(x, y) \). Round all your intermediate calculations to four decimal places and round your final answer to one decimal place.

The closest point on the paraboloid to the point \((1, 1, 1)\) is approximately

\[ x = 2.333 \]
\[ y = -0.473 \]
\[ g_x = 2(x - 2) - 4x(3 - x^2 - y^2) = 0 \]
\[ g_y = 2(y + 3) - 4y(3 - x^2 - y^2) = 0 \]

\[ 2(x - 2) = 4x (\_ \_ \_ ) \Rightarrow \frac{2(x - 2)}{4x} = (\_ \_ \_ ) \]
\[ 2(y + 3) = 4y (\_ \_ \_ ) \Rightarrow \frac{2(y + 3)}{4y} = (\_ \_ \_ ) \]
\[ \frac{2(x - 2)}{4x} = \frac{2(y + 3)}{4y} \]
\[ \frac{1}{2} - \frac{1}{2x} = \frac{1}{2} + \frac{3}{4y} \]
\[ \frac{1}{2x} = \frac{3}{4y} \]
\[ \frac{1}{2} \frac{3}{2} = \frac{3}{4} \frac{1}{4y} \]

\[ y = -6x \]
\[ x = -\frac{3}{2} \]

\[ g_x - g_y = 0 \] numerically yields \((1.033, -1.5495)\).

\[ D(1.033, -1.5495) = 3.2581 \text{ and } g_{xx}(1.033, -1.5495) = 12.4089. \]

Therefore this point is a minimum.

The closest point on the paraboloid to the point \((2, -3, 1)\) is \((1.033, -1.5495, 0.532)\).

Round your answers to three decimal places, if needed.

Find the absolute extrema of the function
\[ f(x, y) = 2.1x^2 - 1.5y^2 - 1.2x - 4.9y \]
on the region bounded by \(x, y \leq 3\), and \(x = 0\).
The absolute maximum is \(x = 1.1442\), \(y = 1.625\).
The absolute minimum is \(x = 2.85\), \(y = 1.43\).
Round your answers to three decimal places, if needed.

Find the absolute extrema of the function

\[ f(x, y) = -2.1x^2 + 1.5y^2 - 1.2x - 4.9y \]

on the region bounded by \( y = x, \ y = 3, \) and \( x = 0. \)

The absolute maximum is \( (\_, \_, \_) \).

The absolute minimum is \( (\_, \_, \_) \).

\[ y = 3.6x^2 - 6.1x \]

\[ (0.847, 0.847) \]

The distance from a point \((x, y)\) to the point \((1, -4)\) is

\[ d(x, y) = \sqrt{(x-2)^2 + (y+4)^2 + (3-x^2-y^2)^2}. \]

To minimize this it is useful to note that we can minimize \( g(x, y) = d(x, y)^2 = D(x, y) \) instead.

\[ g_x = 2(x - 2) - 4x(3 - x^2 - y^2) \]
\[ g_y = 2(y + 4) - 4y(3 - x^2 - y^2) \]
\[ g_{xx} = -10 - 12x^2 + 4y^2 \]
\[ g_{yy} = -10 + 12y^2 + 4x^2 \]
\[ g_{xy} = 8xy \]

\( g_x - g_y = 0 \) numerically yields \((0.8565, -1.7129)\).

\( D(0.8565, -1.7129) = 6.9841 \) and \( g_{xx}(0.8565, -1.7129) = 10.5362 \).

Therefore this point is a minimum.

The closest point on the paraboloid to the point \((2, -4, 1)\) is \( (0.8565, -1.7129, 0.3324) \).
Project:
Get Autocad
(windows or Mac)

Suppose that the sag in a beam of length \( L \), width \( w \) and height \( h \) is given by

\[
S(L, w, h) = 0.0006 \frac{L^3}{wh^3},
\]

with all lengths measured in inches. We illustrate the beam

In the figure below.

A typical beam.
A beam is supposed to measure \( L = 32 \), \( w = 5 \) and \( h = 9 \) with a corresponding sag of 0.1276 inches. Due to weathering and other factors, the manufacturer only guarantees measurements with error tolerances \( L = 32 \pm 2 \), \( w = 5 \pm 0.3 \) and \( h = 9 \pm 0.8 \). Use a linear approximation to estimate the possible range of sag in the beam. Calculate your intermediate calculations and your final answer to five decimal places.

Based on the linear approximation, the sag is 0.3726 ± \[ \text{[value]} \].

\[
dS = \frac{25}{2L} dL + \frac{25}{2w} dw + \frac{25}{3h} dh
\]

\[
+ \frac{4L^3}{3wh^3} dL - \frac{L^4}{wh^5} dw + \frac{3L^4}{wh^5} dh
\]

\[
\text{out: } \frac{L^3}{wh^3} \left( 4 \frac{dL}{w} - \frac{L}{w} dw + \frac{3L}{h} dh \right)
\]

\[
\text{out: } \frac{32^3}{5 \cdot 9} \left( 4 \times 2 + \frac{32}{5} \times 0.3 + \frac{3 \times 32}{9} \times 0.8 \right)
\]
We first compute
\[ \frac{\partial S}{\partial L} = 0.0016 \frac{L^3}{w^2 h^2}, \quad \frac{\partial S}{\partial w} = -0.0004 \frac{L^2}{w^2 h^2} \quad \text{and} \quad \frac{\partial S}{\partial h} = -0.0012 \frac{L^2}{w^2 h^4}. \]

At the point (37, 3, 7), we then have
\[ \frac{\partial S}{\partial L}(37, 3, 7) = 0.07876, \quad \frac{\partial S}{\partial w}(37, 3, 7) = -0.24285 \quad \text{and} \quad \frac{\partial S}{\partial h}(37, 3, 7) = -0.31223. \]

The linear approximation of the sag is then given by
\[ S = 0.7285 + 0.07876(L - 37) - 0.24285(w - 3) - 0.31223(h - 7). \]

From the stated tolerances, \( L - 37 \) must be between \(-1\) and \(1\), \( w - 3 \) must be between \(-0.3\) and \(0.3\) and \( h - 7 \) must be between \(-0.7\) and \(0.7\). Notice that the maximum sag then occurs with \( L - 37 = 1, w - 3 = -0.3\) and \( h - 7 = -0.7\). The linear approximation predicts that
\[ S = 0.7285 \approx 0.07876 + 0.21856 = 0.37018. \]

Similarly, the minimum sag occurs with \( L - 37 = -1, w - 3 = 0.3\) and \( h - 7 = 0.7\). The linear approximation predicts that
\[ S = 0.7285 \approx -0.07876 - 0.21856 = -0.37018. \]

Based on the linear approximation, the sag is \(0.7285 \pm 0.37018\), or between 1.09868 and
Use Lagrange multipliers to find the maximum and minimum of the function $f(x, y)$ subject to the constraint $g(x, y) = c$.

$f(x, y) = x^2y^2$ subject to $x^2 + 4y^2 = 56$

$g(x, y) = x^2 + 4y^2 - 56 - 0$

$\nabla f = \begin{pmatrix} 2xy^2, 2x^2y \end{pmatrix}$

$\nabla g = \begin{pmatrix} 2x, 8y \end{pmatrix}$

$\nabla f = \lambda \nabla g$

$2xy^2 = 2x\lambda$

$2x^2y = 8y\lambda$

Eliminating $\lambda$ we get $y = \pm \frac{1}{2} x$.

$\begin{align*}
\text{Constraint} & \quad g \\
0 &= x^2 + y^2 - z - 3 \\
(x-2)^2 + (y+4)^2 + (z-1)^2 &= 1 \\
\nabla f &= \begin{pmatrix} 2(x-2), 2(y+4), 2(z-1) \end{pmatrix} \\
\nabla g &= \begin{pmatrix} 2x, 2y, -1 \end{pmatrix} \\
\nabla f &= \lambda \nabla g \\
2x-4 &= 2\lambda x \\
2y+8 &= 2\lambda y \\
2z-2 &= -\lambda \\
2 &= x^2 + y^2 - 3
\end{align*}$