Agenda

Lecture Gradient, Directional Derivative

Project: water flowing out of opening in dome

Lecture: Intro to Optimization

\[ \nabla f = \langle f_x, f_y \rangle \]

Find the gradient of the function \( f(x, y) = x^2 + 9xy^2 - y^9 \).

\[ \nabla f = \langle 2x + 9y^2, 18xy - 9y^8 \rangle \]
$\nabla f(0,1)$ is gradient evaluated at $(0,1)$.

$\nabla f(0,1) = \langle 2x + 9y^2, 18xy - 7y^3 \rangle$

$\langle 0 + 9, 0 - 9 \rangle$

$\langle 9, -9 \rangle$

**Directional Derivative**

$D_{\hat{u}}f = \nabla f \cdot \hat{u}$

$\langle fx, fy \rangle \cdot \langle a, b \rangle = afx + bfy$
Compute the directional derivative of \( f \) at the given point in the direction of the indicate vector. 

\[ f(x, y) = e^{4x^2 - y}, \quad (1, 4), \quad \text{u in the direction of} \quad -4i - j \]

\[ D_u f(1, 4) = \] 

\[ \nabla f = \left( e^{4x^2 - y} \cdot \frac{4x^2 - y}{y}, e^{4x^2 - y} \right) \]

\[ \nabla f(1, 4) = \left( 8, -1 \right) \]

\[ D_u f(1, 4) = \nabla f(1, 4) \cdot \mathbf{u} = \frac{\left< 8, -1 \right> \cdot \left< -4, -1 \right>}{\sqrt{17}} \]

\[ \frac{-32 + 1}{\sqrt{17}} = -\frac{31}{\sqrt{17}} \]

\[ \alpha \cdot \mathbf{b} = |a||b| \cos \theta \quad (\theta = 0) \]

Max at \( \cos \theta = 1 \)

\[ \nabla f \cdot \mathbf{u} = |\nabla f| \cdot |\mathbf{u}| = |\nabla f| \]

Max when \( \mathbf{u} \) and \( \nabla f \) 11.
Find the directions of maximum and minimum change of \( f \) at the given point, and the values of the maximum and minimum rates of change.

\[ f(x, y) = 3y^2e^{2x}, \quad (4, 5) \]

\[ \nabla f = \left< 6y^2e^{2x}, 6ye^{2x} \right> \]

\[ \nabla f(4, 5) = \left< 150e^8, 30e^8 \right> \]

\[ \|\nabla f(4, 5)\| = 30e^8\sqrt{26} \]

The maximum change is \( 30e^8\sqrt{26} \); in the direction \( \left< 150e^8, 30e^8 \right> \)

The minimum change is \( -30e^8\sqrt{26} \); in the direction \( \left< -150e^8, -30e^8 \right> \)

Use the table below to estimate \( \nabla f(0, 0) \). Round each component to three decimal places, if needed.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( x - 0.1 )</th>
<th>( x - 0 )</th>
<th>( x - 0.1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y = -0.2 )</td>
<td>2.14</td>
<td>2.36</td>
<td>2.02</td>
</tr>
<tr>
<td>( y = 0 )</td>
<td>1.36</td>
<td>1.96</td>
<td>2.46</td>
</tr>
<tr>
<td>( y = 0.2 )</td>
<td>0.83</td>
<td>1.45</td>
<td>2.08</td>
</tr>
</tbody>
</table>

\[ \nabla f(0, 0) = \left< \frac{-91}{40}, -2.275 \right> \]

Since \( y = 0 \):

\[ \frac{\Delta x}{\Delta y} = \frac{2.36 - 1.36}{0.2 - (-0.2)} = \frac{1}{1} = 5.5 \]

\[ \frac{\Delta y}{\Delta x} = \frac{1.45 - 2.36}{0.2 - (-0.2)} = \frac{-0.9}{0.4} = -2.25 \]
Project: \[ \text{dome } Z(x,y) \]

Which direction will water flow out of hole?

Find \[ \nabla Z \]

\[
Z = 100 - (75 - x - y)^2
\]
\[
\nabla Z = f_x(x,y)i + f_y(x,y)j
\]
\[
\nabla Z = -\frac{x}{\sqrt{75-x-y}} + \frac{y}{\sqrt{75-x-y}}
\]
\[
\nabla Z(5.87, 5.87) = \frac{5.87}{\sqrt{75-5.87}} \frac{5.87}{\sqrt{75-5.87}} = 5.87
\]
\[
\nabla Z(5.87, 5.87) \approx (2.38, 2.38)
\]

Minimum increase of \[ Z = -\|\nabla Z(x,y)\| = -2.38^2 + 2.38^2 = -3.37 \]
The direction of flow is given by the gradient vector at a point. The gradient of a function $Z = f(x, y)$ is defined as:

$$\nabla Z = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right)$$

Given $Z(x, y) = 100 + (75 - x - y)^2$$

The partial derivatives are:

$$\frac{\partial Z}{\partial x} = -2(75 - x - y)$$
$$\frac{\partial Z}{\partial y} = -2(75 - x - y)$$

At the point $(x, y) = (5.87, 5.87)$, the gradient is:

$$\nabla Z(5.87, 5.87) = \left( -2.38, -2.38 \right)$$

The minimum increase of $Z$ is given by the dot product of the gradient and the direction of flow.

$$\text{Direction of flow} = \left( \cos \theta, \sin \theta \right)$$

Projecting the gradient onto the direction of flow gives:

$$\text{Maximum change} = \sqrt{(-0.5)^2 + (0.5)^2} = \sqrt{0.25} = 0.5$$

$$\text{Minimum change} = -0.5$$
Parametric Curves in 3D

Enter formulas for x, y, and z in terms of the parameter t. Use ordinary syntax. For example:

\[ \begin{align*}
  x &= \cos(\pi t) \\
  y &= \sin(\pi t) \\
  z &= t
\end{align*} \]

Mouse over the HELP button for rules and instructions.

x = \cos(\pi t)

y = \sin(\pi t)

z = t

Enter the range for t. (Numerical values and multiple of pi, e.g., 0.5pi)

Lower limit: 0

Upper limit: \pi/2

After you enter or change any of the above settings, click the GRAPH button.

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Optimization (max/min)
Critical Points \( \langle f_x, f_y \rangle = \langle 0, 0 \rangle \)

\[ f = \langle 0, 0 \rangle \]

Constraint Problems:
\[ A = L \times W \quad 2L + W = 10 \]

\( F(x, y) = \)

1. Find critical points \( x, y, \ldots \)
2. Evaluate \( f(x, y) \)
3. Evaluate \( f(x) \)

\( \text{Max = biggest} \)
\( \text{Min = smallest} \)
Locate all critical points and classify them.

\[ f(x, y) = 2x^3 - 6xy + 3y^3 \]

- A. \((0, 0)\) is a local maximum and \((-1, -1)\) is a local minimum.
- B. \((0, 0)\) is a local minimum and \((1, 1)\) is a local maximum.
- C. \((0, 0)\) is a local extremum and \((-1, -1)\) is a saddle point.
- D. \((0, 0)\) is a saddle point and \((1, 1)\) is a local minimum.

\[ f_x = 6x^2 - 6y = 0 \]
\[ f_y = -6x + 6y^2 = 0 \]
\[ f_{xy} f_{yy} - f_{xy}^2 = D \]
\[ D = \left( \frac{12x}{12x} \right)^2 - (-6)^2 = 144x^2 - 36 \]
\[ D > 0 \quad \text{max/min} \]
\[ D < 0 \quad \text{saddle} \]
\[ D = 0 \quad \text{inconclusive} \]

Round your answers to three decimal places, if needed.

Find the absolute extrema of the function

\[ f(x, y) = 2.9x^2 + 3.6y^2 - 3x - 4.1y \]
on the region bounded by \( y = x, y = 3, \) and \( x = 0. \)
The absolute maximum is ____________.
The absolute minimum is ____________.

\[ f_x = 5.8x - 3 = 0 \]
\[ f_y = 7.2y - 4.1 = 0 \]
\[ f(x) = 2.9x^2 + 3.6y^2 - 3x - 4.1y \]
\[ f(x) = 3.6y - 4.1y \]
\[ 0 \leq y \leq 3 \]
\[ f(0) = -3c \quad \text{saddle} \]
\[ f(1, 1) = 12 > 0 \quad \text{min} \]
\[ f(x) = 2.9x^2 + 3.6y^2 - 3x - 4.1y \]
\[ 0 \leq x \leq 3 \]
\[ y = 3 \quad f(x) = 2.9x^2 + 3.6y^2 - 3x - 4.1y \]
\[ 0 \leq y \leq 3 \]
\[ x = 3/5, 8 = 57.1\ldots \]
\[ y = 4.1/2, 2 = 56.1\ldots \]
\[ \text{Check} \]

\[ \text{Ins, Ext} \]

Region.
Find equations of the tangent plane and normal line to the surface at the given point.
\[ z = x^2 + y^3 \] at \((1, -1, 2)\)

\[ N = \langle f_x, f_y, -1 \rangle \]
\[ = \langle 2x, 3y^2, -1 \rangle \] plus in \( x = 1, y = -1, z = 2 \)

\[ N = \langle 2, 3, -1 \rangle \] \( P_0 = (1, -1, 2) \)

\[ 2(x-1) + 3(y+1) - (z-2) = 0 \]
\[ x = 1 - 2t \quad y = -1 + 3t \quad z = 2 - t \]