37. The solid below \( x^2 + y^2 + z^2 = 4z \) and above \( z = \sqrt{x^2 + y^2} \)

\[
\begin{align*}
\rho^2 &= 4 \rho \cos \phi \\
\rho^2 &= 2 \rho \cos \phi \\
x^2 + y^2 + z^2 &= 2z \\
\rho^2 &= 2 \rho \cos \phi \\
\rho^2 (1 - 2 \cos \phi) &= 0
\end{align*}
\]

\[
\int_0^{2\pi} \int_0^{\pi/2} \int_0^{4 \cos \phi} f(x, y, z) \, dV
\]

\[
= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{4 \cos \phi} f(\rho \cos \theta \sin \phi, \rho \sin \theta \sin \phi, \rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

\[
= \int_0^{2\pi} \int_0^{\pi/2} \int_0^{4 \cos \phi} \rho^2 \sin \phi \, d\rho \, d\theta \, d\phi
\]

\[
= \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{3} [4 \cos \phi]_0^{\pi/3} \sin \phi \, d\theta \, d\phi
\]

\[
= \int_0^{2\pi} \int_0^{\pi/2} \frac{1}{3} \sin \phi \, d\theta \, d\phi
\]
52. \[ \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^{\sqrt{x^2+y^2}} \sqrt{x^2 + y^2 + z^2} \, dz \, dy \, dx \]

\[ \rho \quad \phi \quad \theta \]

\[ \rho = \frac{y}{\cos \phi} = 4 \sec \phi \]

\[ \int_0^{\pi/4} \int_0^{\pi/2} \int_0^{\pi/4} p^3 \sin \phi \, \rho \, d\phi \, d\rho \, d\theta \]

\[ \int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\pi/4} p^3 \sin \phi \, \rho \, d\phi \, d\rho \, d\theta \]

\[ (\frac{\pi}{2}) \int_0^{\pi/4} \frac{4 \sec^4 \phi \sin \phi}{\rho} \, d\rho \]
VECTOR FIELDS

For the vector field $\mathbf{F}(x, y) = \langle x + y, -2y + x \rangle$, evaluate $\mathbf{F}(0, 1)$. Plot the vector $\mathbf{F}(x, y)$ using the point $(0, 1)$ as the initial point.

$\langle 0+1, -2+0 \rangle = \langle 1, -2 \rangle$
Gradient Field \( \langle f_x, f_y \rangle \)

\[ \nabla f = \langle f_x, f_y \rangle \]

Find the gradient field corresponding to the function \( f(x, y) = x^3 y - e^y \).

\[ \langle 3x^2y, x^3 - e^y \rangle \]

Gradient Field is also called a Conservative Vector Field
Determine whether or not the vector field \( \langle 17, y \rangle \) is conservative. If it is, find a potential function.

If \( \nabla f(x, y) = \langle 17, y \rangle \), then \( \frac{\partial f}{\partial x} = 17 \) and \( \frac{\partial f}{\partial y} = y \).

\[
\begin{align*}
K(x, y) &= \int 17 \, dx = 17x + g(y) \\
\frac{\partial f}{\partial y} &= g(y) = v \\
g(y) &= \frac{v^2}{2} + c \\
K(x, y) &= 17x + \frac{v^2}{2} + c
\end{align*}
\]

The vector field is conservative.
Determine whether or not the vector field \((x - 28xy)i + (y^2 - 14x^2)j\) is conservative. If it is, find a potential function.

\[
\frac{\partial f}{\partial x} = x - 28xy \\
f = \frac{x^2}{2} - 14xy^2 + g(y) \\
\frac{\partial^2 f}{\partial y^2} = y^2 - 14x^2 \\
f = \frac{y^3}{3} - 14xy^2 + g(x) \\
f = \frac{x^3}{2} - 14xy^2 + \frac{y^3}{3}
\]
A two-dimensional force acts radially away from the origin with magnitude 5. Write the force as a vector field.

\[ \mathbf{F} = \frac{5\mathbf{r}}{||\mathbf{r}||} \]

where \( \mathbf{r} = \langle x, y \rangle \) is the position vector and \( ||\mathbf{r}|| = \sqrt{x^2 + y^2} \).

\[ \langle x, y \rangle \text{ as unit vector} \]

\[ \frac{\langle x, y \rangle}{||\mathbf{r}||} = \langle \frac{x}{\sqrt{x^2 + y^2}}, \frac{y}{\sqrt{x^2 + y^2}} \rangle = \mathbf{u} \]
Compute the exact work done by the force field \( \mathbf{F}(x, y, z) = (2y, 2xz, 6y) \) acting on an object as it moves along the helix defined parametrically by \( x = 4 \cos t \), \( y = 4 \sin t \) and \( z = 6t \), from the point \((4, 0, 0)\) to the point \((-4, 0, 6\pi)\).

\[
\int_{t_1}^{t_2} \mathbf{F} \cdot d\mathbf{r}
\]

\[
\mathbf{r}(t) = (4 \cos t, 4 \sin t, 6t)
\]

\[
\mathbf{F}(t) = (2y, 2xz, 6y)
\]

\[
\int_{0}^{\pi} (-32 \sin^2 t + 192 \cos^2 t + 144 \sin t) \, dt
\]

\[
\frac{6}{6}
\]
Line Integral

single integral over a curve in multiple dimensions

Evaluate the line integral.

\[ \int_C 2x \, ds \], where \( C \) is the line segment from (1, 2) to (7, 7)

\[ x(t) = 1 + 6t \]
\[ y(t) = 2 + 5t \]

\[ ds = \sqrt{(x')^2 + (y')^2} \, dt \]
\[ = \sqrt{(6)^2 + (5)^2} \, dt \]
\[ = \sqrt{61} \, dt \]

\[ \int_0^1 2(1 + 6t) \sqrt{61} \, dt = 2 \sqrt{61} \left[ t + 3t^2 \right]_0^1 = 8 \sqrt{61} \]
Evaluate the line integral.

\[ \int_{C} 6z \, ds, \text{ where } C \text{ is the line segment from } (5, 0, 1) \text{ to } (6, -10, 10) \]

\[ \int_{0}^{1} 6(1+9t) \sqrt{1+81} \, dt \]

\[ ds = \sqrt{1+100+81} \, dt = \sqrt{182} \, dt \]

\[ x = 5 + t \]
\[ y = 0 - 10t \]
\[ z = 1 + 9t \]

\[ 0 \leq t \leq 1 \]
Compute the work done by the force field $F$ along the curve $C$.

$F(x, y) = \langle 9x, 9y \rangle$, $C$ is the line segment from $(1, 1)$ to $(3, 4)$

$$x = 1 + 2t, \quad y = 1 + 3t, \quad 0 \leq t \leq 1$$

$$\int_C F \cdot dr = \int_C 9x \, dx + 9y \, dy$$

$$= \int_0^1 [9(1 + 2t)(2) + 9(1 + 3t)(3)] \, dt$$

$$= \int_0^1 (117t + 45) \, dt = \frac{207}{2}$$