1. In each case verify that the following are solutions for all values of s and t.

(a) $x = 19t - 35$
$y = 25 - 13t$
$z = t$
is a solution of
$2x + 3y + z = 5$
$5x + 7y - 4z = 0$

•[2](b) $x_1 = 2s + 12t + 13$
$x_2 = s$
$x_3 = -s - 3t - 3$
$x_4 = t$
is a solution of
$2x_1 + 5x_2 + 9x_3 + 3x_4 = -1$
$x_1 + 2x_2 + 4x_3 = 1$

2. Find all solutions to the following in parametric form in two ways.

(a) $3x + y = 2$ (b) $2x + 3y = 1$

(c) $3x - y + 2z = 5$ (b) $x - 2y + 5z = 1$

3. Regarding $2x = 5$ as the equation $2x + 0y = 5$ in two variables, find all solutions in parametric form.

•4. Regarding $4x - 2y = 3$ as the equation $4x - 2y + 0z = 3$ in three variables, find all solutions in parametric form.

•5. Find all solutions to the general system $ax = b$ of one equation in one variable (a) when $a = 0$ and (b) when $a \neq 0$.

6. Show that a system consisting of exactly one linear equation can have no solution, one solution, or infinitely many solutions. Give examples.

7. Write the augmented matrix for each of the following systems of linear equations.

(a) $x - 3y = 5$
$2x + y = 1$

•(b) $x + 2y = 0$
$y = 1$

(c) $x - y + z = 2$
$x - z = 1$
$y + 2x = 0$

•(d) $x + y = 1$
$y + z = 0$
$z - x = 2$

8. Write a system of linear equations that has each of the following augmented matrices.

(a) $\left[\begin{array}{ccc|c} 1 & -1 & 6 & 0 \\ 0 & 1 & 0 & 3 \\ 2 & -1 & 0 & 1 \end{array}\right]$ •(b) $\left[\begin{array}{ccc|c} 2 & -1 & 0 & -1 \\ -3 & 2 & 1 & 0 \\ 0 & 1 & 1 & 3 \end{array}\right]$

9. Find the solution of each of the following systems of linear equations using augmented matrices.

(a) $\begin{aligned} x - 3y &= 1 \\ 2x - 7y &= 3 \end{aligned}$ •(b) $\begin{aligned} x + 2y &= 1 \\ 3x + 4y &= -1 \end{aligned}$

(c) $\begin{aligned} 2x + 3y &= -1 \\ 3x + 4y &= 2 \end{aligned}$ •(d) $\begin{aligned} 3x + 4y &= 1 \\ 4x + 5y &= -3 \end{aligned}$

10. Find the solution of each of the following systems of linear equations using augmented matrices.

(a) $\begin{aligned} x + y + 2z &= -1 \\ 2x + y + 3z &= 0 \\ -2y + z &= 2 \end{aligned}$ •(b) $\begin{aligned} 2x + y + z &= -1 \\ x + 2y + z &= 0 \\ 3x - 2z &= 5 \end{aligned}$

11. Find all solutions (if any) of the following systems of linear equations.

(a) $\begin{aligned} 3x - 2y &= 5 \\ -12x + 8y &= -20 \end{aligned}$ •(b) $\begin{aligned} 3x - 2y &= 5 \\ -12x + 8y &= 16 \end{aligned}$

12. Show that the system $\begin{cases} x + 2y - z = a \\ 2x + y + 3z = b \\ x - 4y + 9z = c \end{cases}$ is inconsistent unless $c = 2b - 3a$.

13. By examining the possible positions of lines in the plane, show that two equations in two variables can have zero, one, or infinitely many solutions.

14. In each case either show that the statement is true, or give an example[3] showing it is false.

(a) If a linear system has n variables and m equations, then the augmented matrix has n rows.

•(b) A consistent linear system must have infinitely many solutions.

(c) If a row operation is done to a consistent linear system, the resulting system must be consistent.

•(d) If a series of row operations on a linear system results in an inconsistent system, the original system is inconsistent.

15. Find a quadratic $a + bx + cx^2$ such that the graph of $y = a + bx + cx^2$ contains each of the points $(-1, 6)$, $(2, 0)$, and $(3, 2)$.

•16. Solve the system $\begin{cases} 3x + 2y = 5 \\ 7x + 5y = 1 \end{cases}$ by changing variables $\begin{cases} x = 5x' - 2y' \\ y = -7x' + 3y' \end{cases}$ and solving the resulting equations for x' and y'.

•17. Find a, b, and c such that

$$\frac{x^2 - x + 3}{(x^2 + 2)(2x - 1)} = \frac{ax + b}{x^2 + 2} + \frac{c}{2x - 1}$$

[*Hint*: Multiply through by $(x^2 + 2)(2x - 1)$ and equate coefficients of powers of x.]

18. A zookeeper wants to give an animal 42 mg of vitamin A and 65 mg of vitamin D per day. He has two supplements: the first contains 10% vitamin A and 25% vitamin D; the second contains 20% vitamin A and 25% vitamin D. How much of each supplement should he give the animal each day?

•19. Workmen John and Joe earn a total of \$24.60 when John works 2 hours and Joe works 3 hours. If John works 3 hours and Joe works 2 hours, they get \$23.90. Find their hourly rates.

20. A biologist wants to create a diet from fish and meal containing 183 grams of protein and 93 grams of carbohyrate per day. If fish contains 70% protein and 10% carbohydrate, and meal contains 30% protein and 60% carbohydrate, how much of each food is required each day?

[3] Such an example is called a **counterexample**. For example, if the statement is that "all philosophers have beards", the existence of a non-bearded philosopher would be a counterexample proving that the statement is false. This is discussed again in Appendix B.

SECTION 1.2 Gaussian Elimination

The algebraic method introduced in the preceding section can be summarized as follows: Given a system of linear equations, use a sequence of elementary row operations to carry the augmented matrix to a "nice" matrix (meaning that the corresponding equations are easy to solve). In Example 3 Section 1.1, this nice matrix took the form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right]$$

The following definitions identify the nice matrices that arise in this process.

Definition 1.3 *A matrix is said to be in* **row-echelon form** *(and will be called a* **row-echelon matrix***) if it satisfies the following three conditions:*

1. ***All* zero rows** *(consisting entirely of zeros) are at the bottom.*
2. *The first nonzero entry from the left in each nonzero row is a 1, called the* **leading 1** *for that row.*
3. *Each leading 1 is to the right of all leading 1s in the rows above it.*

A row-echelon matrix is said to be in **reduced row-echelon form** *(and will be called a* **reduced row-echelon matrix***) if, in addition, it satisfies the following condition:*

4. *Each leading 1 is the only nonzero entry in its column.*

The row-echelon matrices have a "staircase" form, as indicated by the following example (the asterisks indicate arbitrary numbers).

$$\begin{bmatrix} 0 & 1 & * & * & * & * & * \\ 0 & 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The leading 1s proceed "down and to the right" through the matrix. Entries above and to the right of the leading 1s are arbitrary, but all entries below and to the left of them are zero. Hence, a matrix in row-echelon form is in reduced form if, in addition, the entries directly above each leading 1 are all zero. Note that a matrix in row-echelon form can, with a few more row operations, be carried to reduced form (use row operations to create zeros above each leading one in succession, beginning from the right).

The following matrices are in row-echelon form (for any choice of numbers in $*$-positions).

$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

Many important problems involve **linear inequalities** rather than **linear equations**. For example, a condition on the variables x and y might take the form of an inequality $2x - 5y \leq 4$ rather than an equality $2x - 5y = 4$. There is a technique (called the **simplex algorithm**) for finding solutions to a system of such inequalities that maximizes a function of the form $p = ax + by$ where a and b are fixed constants. This procedure involves gaussian elimination techniques, and the interested reader can find an introduction on Connect by visiting www.mcgrawhill.ca/college/nicholson and then selecting this text.

EXERCISES 1.2

1. Which of the following matrices are in reduced row-echelon form? Which are in row-echelon form?

(a) $\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ •(b) $\begin{bmatrix} 2 & 1 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & -2 & 3 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ •(d) $\begin{bmatrix} 1 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ •(f) $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

2. Carry each of the following matrices to reduced row-echelon form.

(a) $\begin{bmatrix} 0 & -1 & 2 & 1 & 2 & 1 & -1 \\ 0 & 1 & -2 & 2 & 7 & 2 & 4 \\ 0 & -2 & 4 & 3 & 7 & 1 & 0 \\ 0 & 3 & -6 & 1 & 6 & 4 & 1 \end{bmatrix}$

•(b) $\begin{bmatrix} 0 & -1 & 3 & 1 & 3 & 2 & 1 \\ 0 & -2 & 6 & 1 & -5 & 0 & -1 \\ 0 & 3 & -9 & 2 & 4 & 1 & -1 \\ 0 & 1 & -3 & -1 & 3 & 0 & 1 \end{bmatrix}$

3. The augmented matrix of a system of linear equations has been carried to the following by row operations. In each case solve the system.

(a) $\left[\begin{array}{cccccc|c} 1 & 2 & 0 & 3 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

•(b) $\left[\begin{array}{cccccc|c} 1 & -2 & 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 5 & 0 & -3 & -1 \\ 0 & 0 & 0 & 0 & 1 & 6 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

(c) $\left[\begin{array}{ccccc|c} 1 & 2 & 1 & 3 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

•(d) $\left[\begin{array}{ccccc|c} 1 & -1 & 2 & 4 & 6 & 2 \\ 0 & 1 & 2 & 1 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$

4. Find all solutions (if any) to each of the following systems of linear equations.

(a) $\begin{aligned} x - 2y &= 1 \\ 4y - x &= -2 \end{aligned}$ •(b) $\begin{aligned} 3x - y &= 0 \\ 2x - 3y &= 1 \end{aligned}$

(c) $\begin{aligned} 2x + y &= 5 \\ 3x + 2y &= 6 \end{aligned}$ •(d) $\begin{aligned} 3x - y &= 2 \\ 2y - 6x &= -4 \end{aligned}$

(e) $\begin{aligned} 3x - y &= 4 \\ 2y - 6x &= 1 \end{aligned}$ •(f) $\begin{aligned} 2x - 3y &= 5 \\ 3y - 2x &= 2 \end{aligned}$

5. Find all solutions (if any) to each of the following systems of linear equations.

(a) $\begin{aligned} x + y + 2z &= 8 \\ 3x - y + z &= 0 \\ -x + 3y + 4z &= -4 \end{aligned}$ •(b) $\begin{aligned} -2x + 3y + 3z &= -9 \\ 3x - 4y + z &= 5 \\ -5x + 7y + 2z &= -14 \end{aligned}$

(c) $\begin{aligned} x + y - z &= 10 \\ -x + 4y + 5z &= -5 \\ x + 6y + 3z &= 15 \end{aligned}$ •(d) $\begin{aligned} x + 2y - z &= 2 \\ 2x + 5y - 3z &= 1 \\ x + 4y - 3z &= 3 \end{aligned}$

(e) $\begin{aligned} 5x + y &= 2 \\ 3x - y + 2z &= 1 \\ x + y - z &= 5 \end{aligned}$ •(f) $\begin{aligned} 3x - 2y + z &= -2 \\ x - y + 3z &= 5 \\ -x + y + z &= -1 \end{aligned}$

(g) $\begin{aligned} x + y + z &= 2 \\ x + z &= 1 \\ 2x + 5y + 2z &= 7 \end{aligned}$ •(h) $\begin{aligned} x + 2y - 4z &= 10 \\ 2x - y + 2z &= 5 \\ x + y - 2z &= 7 \end{aligned}$

6. Express the last equation of each system as a sum of multiples of the first two equations. [*Hint*: Label the equations, use the gaussian algorithm.]

(a) $\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 - x_2 + 3x_3 &= 3 \\ x_1 - 2x_2 + 2x_3 &= 2 \end{aligned}$

•(b) $\begin{aligned} x_1 + 2x_2 - 3x_3 &= -3 \\ x_1 + 3x_2 - 5x_3 &= 5 \\ x_1 - 2x_2 + 5x_3 &= -35 \end{aligned}$

7. Find all solutions to the following systems.

(a) $\begin{aligned} 3x_1 + 8x_2 - 3x_3 - 14x_4 &= 2 \\ 2x_1 + 3x_2 - x_3 - 2x_4 &= 1 \\ x_1 - 2x_2 + x_3 + 10x_4 &= 0 \\ x_1 + 5x_2 - 2x_3 - 12x_4 &= 1 \end{aligned}$

•(b) $\begin{aligned} x_1 - x_2 + x_3 - x_4 &= 0 \\ -x_1 + x_2 + x_3 + x_4 &= 0 \\ x_1 + x_2 - x_3 + x_4 &= 0 \\ x_1 + x_2 + x_3 + x_4 &= 0 \end{aligned}$

(c) $\begin{aligned} x_1 - x_2 + x_3 - 2x_4 &= 1 \\ -x_1 + x_2 + x_3 + x_4 &= -1 \\ -x_1 + 2x_2 + 3x_3 - x_4 &= 2 \\ x_1 - x_2 + 2x_3 + x_4 &= 1 \end{aligned}$

•(d) $\begin{aligned} x_1 + x_2 + 2x_3 - x_4 &= 4 \\ 3x_2 - x_3 + 4x_4 &= 2 \\ x_1 + 2x_2 - 3x_3 + 5x_4 &= 0 \\ x_1 + x_2 - 5x_3 + 6x_4 &= -3 \end{aligned}$

8. In each of the following, find (if possible) conditions on a and b such that the system has no solution, one solution, and infinitely many solutions.

(a) $\begin{aligned} x - 2y &= 1 \\ ax + by &= 5 \end{aligned}$ •(b) $\begin{aligned} x + by &= -1 \\ ax + 2y &= 5 \end{aligned}$

(c) $\begin{aligned} x - by &= -1 \\ x + ay &= 3 \end{aligned}$ •(d) $\begin{aligned} ax + y &= 1 \\ 2x + y &= b \end{aligned}$

9. In each of the following, find (if possible) conditions on a, b, and c such that the system has no solution, one solution, or infinitely many solutions.

(a) $\begin{aligned} 3x + y - z &= a \\ x - y + 2z &= b \\ 5x + 3y - 4z &= c \end{aligned}$ •(b) $\begin{aligned} 2x + y - z &= a \\ 2y + 3z &= b \\ x - z &= c \end{aligned}$

(c) $\begin{aligned} -x + 3y + 2z &= -8 \\ x + z &= 2 \\ 3x + 3y + az &= b \end{aligned}$ •(d) $\begin{aligned} x + ay &= 0 \\ y + bz &= 0 \\ z + cx &= 0 \end{aligned}$

(e) $\begin{aligned} 3x - y + 2z &= 3 \\ x + y - z &= 2 \\ 2x - 2y + 3z &= b \end{aligned}$

•(f) $\begin{aligned} x + ay - z &= 1 \\ -x + (a-2)y + z &= -1 \\ 2x + 2y + (a-2)z &= 1 \end{aligned}$

•10. Find the rank of each of the matrices in Exercise 1.

11. Find the rank of each of the following matrices.

(a) $\begin{bmatrix} 1 & 1 & 2 \\ 3 & -1 & 1 \\ -1 & 3 & 4 \end{bmatrix}$ •(b) $\begin{bmatrix} -2 & 3 & 3 \\ 3 & -4 & 1 \\ -5 & 7 & 2 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & -4 \end{bmatrix}$ •(d) $\begin{bmatrix} 3 & -2 & 1 & -2 \\ 1 & -1 & 3 & 5 \\ -1 & 1 & 1 & -1 \end{bmatrix}$

(e) $\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & a & 1-a & a^2+1 \\ 1 & 2-a & -1 & -2a^2 \end{bmatrix}$

•(f) $\begin{bmatrix} 1 & 1 & 2 & a^2 \\ 1 & 1-a & 2 & 0 \\ 2 & 2-a & 6-a & 4 \end{bmatrix}$

12. Consider a system of linear equations with augmented matrix A and coefficient matrix C. In each case either prove the statement or give an example showing that it is false.

(a) If there is more than one solution, A has a row of zeros.

•(b) If A has a row of zeros, there is more than one solution.

(c) If there is no solution, the row-echelon form of C has a row of zeros.

•(d) If the row-echelon form of C has a row of zeros, there is no solution.

(e) There is no system that is inconsistent for every choice of constants.

•(f) If the system is consistent for some choice of constants, it is consistent for every choice of constants.

Now assume that the augmented matrix A has 3 rows and 5 columns.

(g) If the system is consistent, there is more than one solution.

•(h) The rank of A is at most 3.

(i) If rank $A = 3$, the system is consistent.

(j) If rank $C = 3$, the system is consistent.

13. Find a sequence of row operations carrying

$$\begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ c_1+a_1 & c_2+a_2 & c_3+a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} \text{ to } \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

14. In each case, show that the reduced row-echelon form is as given.

(a) $\begin{bmatrix} p & 0 & a \\ b & 0 & 0 \\ q & c & r \end{bmatrix}$ with $abc \neq 0$; $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

•(b) $\begin{bmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{bmatrix}$ where $c \neq a$ or $b \neq a$; $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$

15. Show that $\begin{cases} ax + by + cz = 0 \\ a_1x + b_1y + c_1z = 0 \end{cases}$ always has a solution other than $x = 0, y = 0, z = 0$.

16. Find the circle $x^2 + y^2 + ax + by + c = 0$ passing through the following points.

(a) $(-2, 1)$, $(5, 0)$, and $(4, 1)$

•(b) $(1, 1)$, $(5, -3)$, and $(-3, -3)$

17. Three Nissans, two Fords, and four Chevrolets can be rented for $106 per day. At the same rates two Nissans, four Fords, and three Chevrolets cost $107 per day, whereas four Nissans, three Fords, and two Chevrolets cost $102 per day. Find the rental rates for all three kinds of cars.

•18. A school has three clubs and each student is required to belong to exactly one club. One year the students switched club membership as follows:

Club A. $\frac{4}{10}$ remain in A, $\frac{1}{10}$ switch to B, $\frac{5}{10}$ switch to C.

Club B. $\frac{7}{10}$ remain in B, $\frac{2}{10}$ switch to A, $\frac{1}{10}$ switch to C.

Club C. $\frac{6}{10}$ remain in C, $\frac{2}{10}$ switch to A, $\frac{2}{10}$ switch to B.

If the fraction of the student population in each club is unchanged, find each of these fractions.

19. Given points (p_1, q_1), (p_2, q_2), and (p_3, q_3) in the plane with p_1, p_2, and p_3 distinct, show that they lie on some curve with equation $y = a + bx + cx^2$. [*Hint*: Solve for a, b, and c.]

20. The scores of three players in a tournament have been lost. The only information available is the total of the scores for players 1 and 2, the total for players 2 and 3, and the total for players 3 and 1.

(a) Show that the individual scores can be rediscovered.

(b) Is this possible with four players (knowing the totals for players 1 and 2, 2 and 3, 3 and 4, and 4 and 1)?

21. A boy finds $1.05 in dimes, nickels, and pennies. If there are 17 coins in all, how many coins of each type can he have?

22. If a consistent system has more variables than equations, show that it has infinitely many solutions. [*Hint*: Use Theorem 2.]

$$\text{Hence basic solutions are } \mathbf{x}_1 = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x}_2 = \begin{bmatrix} -2 \\ 0 \\ -6 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \mathbf{x}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

EXERCISES 1.3

1. Consider the following statements about a system of linear equations with augmented matrix A. In each case either prove the statement or give an example for which it is false.
   (a) If the system is homogeneous, every solution is trivial.
   •(b) If the system has a nontrivial solution, it cannot be homogeneous.
   (c) If there exists a trivial solution, the system is homogeneous.
   •(d) If the system is consistent, it must be homogeneous.

   Now assume that the system is homogeneous.

   (e) If there exists a nontrivial solution, there is no trivial solution.
   (f) If there exists a solution, there are infinitely many solutions.
   (g) If there exist nontrivial solutions, the row-echelon form of A has a row of zeros.
   •(h) If the row-echelon form of A has a row of zeros, there exist nontrivial solutions.
   (i) If a row operation is applied to the system, the new system is also homogeneous.

2. In each of the following, find all values of a for which the system has nontrivial solutions, and determine all solutions in each case.

   (a) $\begin{aligned} x - 2y + z &= 0 \\ x + ay - 3z &= 0 \\ -x + 6y - 5z &= 0 \end{aligned}$ •(b) $\begin{aligned} x + 2y + z &= 0 \\ x + 3y + 6z &= 0 \\ 2x + 3y + az &= 0 \end{aligned}$

   (c) $\begin{aligned} x + y - z &= 0 \\ ay - z &= 0 \\ x + y + az &= 0 \end{aligned}$ •(d) $\begin{aligned} ax + y + z &= 0 \\ x + y - z &= 0 \\ x + y + az &= 0 \end{aligned}$

3. Let $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{z} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. In each case, either write $\mathbf{v}$ as a linear combination of $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, or show that it is not such a linear combination.

   (a) $\mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ -3 \end{bmatrix}$ •(b) $\mathbf{v} = \begin{bmatrix} 4 \\ 3 \\ -4 \end{bmatrix}$

   (c) $\mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ •(d) $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$

4. In each case, either express $\mathbf{y}$ as a linear combination of $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$, or show that it is not such a linear combination. Here:

   $\mathbf{a}_1 = \begin{bmatrix} -1 \\ 3 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix}$, and $\mathbf{a}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

   (a) $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 0 \end{bmatrix}$ (b) $\mathbf{x} = \begin{bmatrix} -1 \\ 9 \\ 2 \\ 6 \end{bmatrix}$

5. For each of the following homogeneous systems, find a set of basic solutions and express the general solution as a linear combination of these basic solutions.

   (a) $\begin{aligned} x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 + 2x_3 + x_5 &= 0 \\ 2x_1 + 4x_2 - 2x_3 + 3x_4 + x_5 &= 0 \end{aligned}$

   •(b) $\begin{aligned} x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\ -x_1 - 2x_2 + 2x_3 + x_5 &= 0 \\ -x_1 - 2x_2 + 3x_3 + x_4 + 3x_5 &= 0 \end{aligned}$

   (c) $\begin{aligned} x_1 + x_2 - x_3 + 2x_4 + x_5 &= 0 \\ x_1 + 2x_2 - x_3 + x_4 + x_5 &= 0 \\ 2x_1 + 3x_2 - x_3 + 2x_4 + x_5 &= 0 \\ 4x_1 + 5x_2 - 2x_3 + 5x_4 + 2x_5 &= 0 \end{aligned}$

   •(d) $\begin{aligned} x_1 + x_2 - 2x_3 - 2x_4 + 2x_5 &= 0 \\ 2x_1 + 2x_2 - 4x_3 - 4x_4 + x_5 &= 0 \\ x_1 - x_2 + 2x_3 + 4x_4 + x_5 &= 0 \\ -2x_1 - 4x_2 + 8x_3 + 10x_4 + x_5 &= 0 \end{aligned}$

6. (a) Does Theorem 1 imply that the system $\begin{cases} -z + 3y = 0 \\ 2x - 6y = 0 \end{cases}$ has nontrivial solutions? Explain.
   •(b) Show that the converse to Theorem 1 is not true. That is, show that the existence of nontrivial solutions does *not* imply that there are more variables than equations.

7. In each case determine how many solutions (and how many parameters) are possible for a homogeneous system of four linear equations in six variables with augmented matrix A. Assume that A has nonzero entries. Give all possibilities.
   (a) Rank $A = 2$.
   •(b) Rank $A = 1$.
   (c) A has a row of zeros.
   •(d) The row-echelon form of A has a row of zeros.

8. The graph of an equation $ax + by + cz = 0$ is a plane through the origin (provided that not all of a, b, and c are zero). Use Theorem 1 to show that two planes through the origin have a point in common other than the origin $(0, 0, 0)$.

9. (a) Show that there is a line through any pair of points in the plane. [*Hint*: Every line has equation $ax + by + c = 0$, where a, b, and c are not all zero.]
   •(b) Generalize and show that there is a plane $ax + by + cz + d = 0$ through any three points in space.

10. The graph of $a(x^2 + y^2) + bx + cy + d = 0$ is a circle if $a \neq 0$. Show that there is a circle through any three points in the plane that are not all on a line.

•11. Consider a homogeneous system of linear equations in n variables, and suppose that the augmented matrix has rank r. Show that the system has nontrivial solutions if and only if $n > r$.

12. If a consistent (possibly nonhomogeneous) system of linear equations has more variables than equations, prove that it has more than one solution.