3. VECTOR SPACES AND SUBSPACES (3.5 weeks) = chpt 5 & chpt 6
At the end of Unit 3, the student should be able to:
- ☐ explain the defining properties of a vector space. (Course Goal 3)
- ☐ construct examples of vector spaces. (Course Goal 3)
- ☐ explain why a set defined with the necessary operations is or is not a vector space. (Course Goal 3)
- ☐ explain why a subset of a given vector space is or is not a subspace. (Course Goal 3)
- ☐ define the span of a set of vectors. (Course Goal 3)
- ☐ determine if a collection of vectors from a given vector space is a spanning set for the vector space. (Course Goal 3, 12)
- ☐ define linear independence. (Course Goal 3 and 5)
- ☐ calculate whether or not a given set of vectors is linearly independent. (Course Goal 5,12)
- ☐ calculate the rank of a given matrix. (Course Goal 5, 12)
- ☐ explain the defining properties of a basis for a vector space. (Course Goal 5)
- ☐ determine if a given set of vectors from a vector space is or is not a basis for the space. (Course Goals 5, 12)
- ☐ explain what is meant by the dimension of a vector space. (Course Goal 5)MAT208 Course Outline Fall 2012 4
- ☐ state the properties of subspaces and the relationships among the four fundamental subspaces of a matrix. (Course Goals 3 and 5, Gen Ed Goals 1 and 2)
- ☐ explain why the equation Ax=b is consistent if and only if b is in the column space of A. (Course Goals 3 and 5, Gen Ed Goals 1 and 2)
- ☐ discuss how linear independence, spanning sets, basis and dimension are related. (Course Goals 5

vector space: 0, aX, X+Y

4. ORTHOGONALITY AND LEAST SQUARES (2 weeks) = Chpater 5 & 8
At the end of Unit 4, the student should be able to:
- ☐ state the properties of orthogonal matrices. (Course Goal 7)
- ☐ derive the normal equations for a least squares problem and solve it. (Course Goal 7)
- ☐ explain what condition must be satisfied for the normal equations to have a unique solution. (Course Goal 6)
- ☐ apply least squares approximations to problems to minimize errors. (Course Goals 6 and 12)
- ☐ calculate the error in a least squares problem. (Course Goals 6 and 12)
- ☐ apply the Gram-Schmidt process to construct an orthonormal set of vectors. (Course Goal 7, 12)
- ☐ construct the QR factorization of a matrix. (Course Goal 7 and 12)
- ☐ apply the QR factorization to least-squares problem. (Course Goals 6,7, and 12)
- ☐ compare the methods of matrix factorization studied so far. (Course Goals 2 and 7)

5. EIGENVALUES AND EIGENVECTORS (3 weeks) = chapter 8
At the end of Unit 5, the student should be able to:
☐ define eigenvalue and eigenvector. (Course Goal 10)
☐ calculate the characteristic equation of a square matrix and solve to find eigenvalues. (Course Goal 10, 12)
☐ calculate the associated eigenvectors of a square matrix. (Course Goal 10, 12)
☐ explain how the eigenvalues of similar matrices are related. (Course Goals 4 and 10)
☐ analyze the relationship among determinants and number of eigenvalues, determinants and the product of eigenvalues, and the trace and the sum of eigenvalues. (Course Goals 4, 10, 12)
☐ explain the application of diagonalizing a matrix to the matrix exponential. (Course Goal 9)
☐ define pseudoinverse for a non-square matrix. (Course Goal 11)
☐ construct the singular value decomposition of an mxn matrix. (Course Goal 9, 12)
☐ explain the relationships among the columns of U and V of
T A U V ☐ ☐
, the singular value
decomposition of an mxn matrix A, and the four fundamental subspaces associated with A. (Course Goals 5, 9, 10, 12)
☐ summarize the use of the singular value decomposition in applications such as Web search engines and image processing. (Course Goal 9, 12)
☐ apply the singular value decomposition and pseudoinverse to solve least squares problems. (Course Goals 9,11, and 12)
☐ conclude, through the use of technology, that the singular value decomposition is necessary for factoring non-square matrices. (Course Goals 9 and 12)MAT208 Course Outline Fall 2012

Four Fundamental Subspaces of a matrix

name of subspace	definition	containing space	dimension	basis
column space, range or image	$\mathrm{im}(A)$ or $\mathrm{range}(A)$	$\mathbf{R}^m$	r (rank)	The first r columns of U
nullspace or kernel	$\ker(A)$ or $\mathrm{null}(A)$	$\mathbf{R}^n$	$n-r$ (nullity)	The last $(n-r)$ columns of V
row space or coimage	$\mathrm{im}(A^{\mathrm{T}})$ or $\mathrm{range}(A^{\mathrm{T}})$	$\mathbf{R}^n$	r (rank)	The first r columns of V
left nullspace or cokernel	$\ker(A^{\mathrm{T}})$ or $\mathrm{null}(A^{\mathrm{T}})$	$\mathbf{R}^m$	$m-r$ (corank)	The last $(m-r)$ columns of U

Axiom	Meaning
Associativity of addition	$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$
Commutativity of addition	$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
Identity element of addition	There exists an element $\mathbf{0} \in V$, called the *zero vector*, such that $\mathbf{v} + \mathbf{0} = \mathbf{v}$ for all $\mathbf{v} \in V$.
Inverse elements of addition	For every $\mathbf{v} \in \mathrm{V}$, there exists an element $-\mathbf{v} \in V$, called the *additive inverse* of $\mathbf{v}$, such that $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$.
Compatibility of scalar multiplication with field multiplication	$a(b\mathbf{v}) = (ab)\mathbf{v}$ [nb 2]
Identity element of scalar multiplication	$1\mathbf{v} = \mathbf{v}$, where 1 denotes the multiplicative identity in F.
Distributivity of scalar multiplication with respect to vector addition	$a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$
Distributivity of scalar multiplication with respect to field addition	$(a + b)\mathbf{v} = a\mathbf{v} + b\mathbf{v}$

vectors same size R2 R3

$x = 0 \quad y = -z$

$a\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

Failure :(

x=0 y = 1

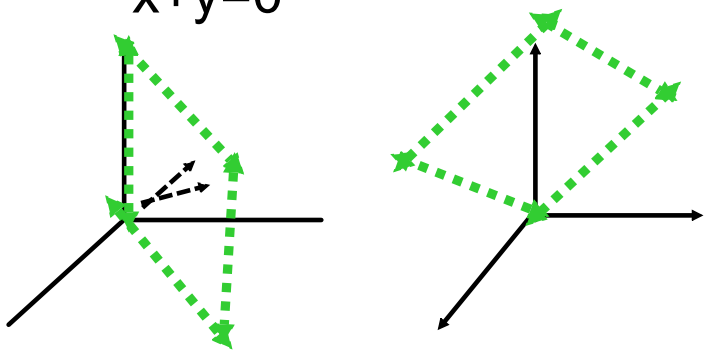
x=y+1

$x=y^2$

Subsets of R3

Plane: Examples x = 3 (NOT A SUBSPACE)

Example: x+y=0



Example: sphere of radius 4

span of a set of a set of vectors

is the set of all linear combinations of the vectors

span (v1, v2) = aV1+bV2

v1,v2,v3 in R3

might span R3 if they are linearly independent

Geometric: Not all on same plane (or in same line)

v1,v2,v3,v4 span R4?

a1v1+a2v2+a3v3+a4v4=0

a1=a2=a3=a4=0

$$\text{rref}\left[\begin{array}{cccc|c} v_1 & v_2 & v_3 & v_4 & \begin{matrix}0\\0\\0\\0\end{matrix}\end{array}\right] \to \left[\begin{array}{c|c} I & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 3 & \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\overset{-5.5}{(1,2,3)},\overset{3}{(4,5,8)},\overset{-.5}{(9,0,1)}$$

$$\text{rref}\begin{bmatrix} 1 & 4 & 9 \\ 2 & 5 & 0 \\ 3 & 8 & 1 \end{bmatrix} = I$$

$$\text{rref}\begin{bmatrix} 1 & 4 & 9 & 2 \\ 2 & 5 & 0 & 4 \\ 3 & 8 & 1 & 7 \end{bmatrix}$$

(2,4,7)

$$\begin{bmatrix} 1 & 4 & 9 \\ 2 & 5 & 0 \\ 3 & 8 & 1 \end{bmatrix} \begin{matrix} a \\ b \\ c \end{matrix} = \begin{matrix} x \\ y \\ z \end{matrix}$$

rank = # of pivot columns

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

rank = 3

basis for the column space:

v1, v2, v4 or v1,v3,v4 or v2,v3,v4

basis for a vector space:

minimum number of vectors needed to span the space.

dimension of a vector space =

number of vectors in the basis

Ax= b

$$\underset{m \times n}{[v1\ v2 \ldots vn]} \underset{n \times 1}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \underset{m \times 1}{\begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}}$$

x1v1+x2V2+.... xnVn= b