Merlin’s Magic Squares

MAT208

FALL 2009

The Game

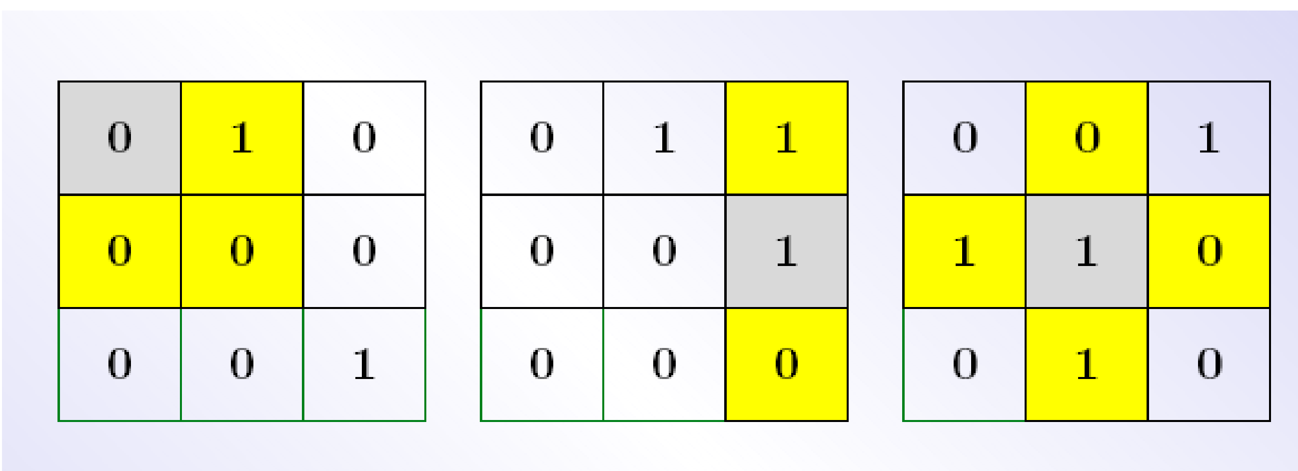
Merlin's Magic Squares is a hand held electronic game made by Parker Brothers. It consists of a 3x3 array with nine boxes, each containing either a one or a zero.

1	0	0
1	1	0
0	0	1

The Rules

1	0	0
1	1	0
0	0	1

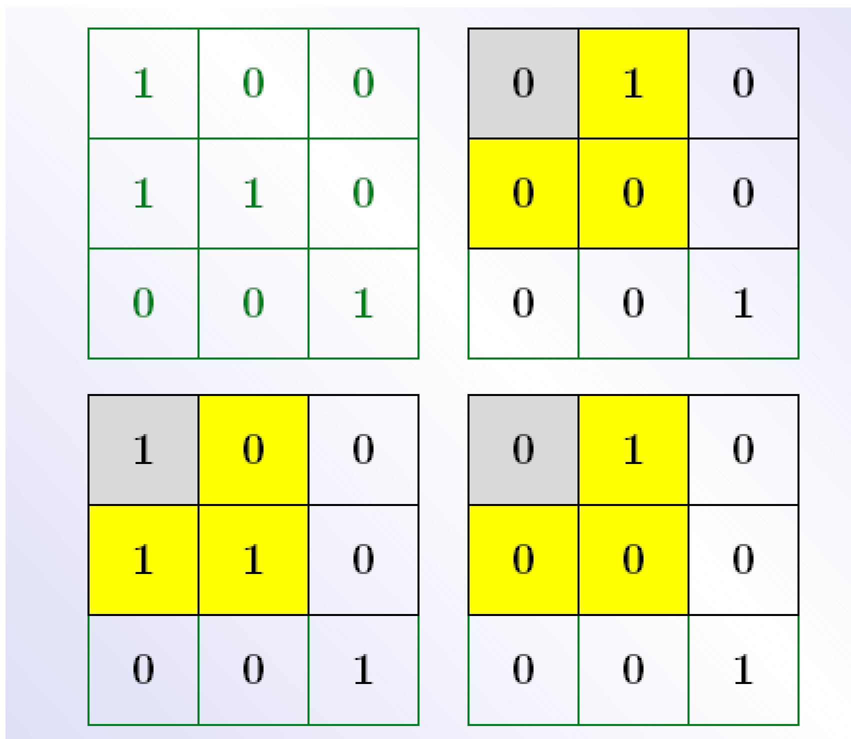
When a button is pressed, the button and those that surround it are changed to their opposite states in the following patterns:



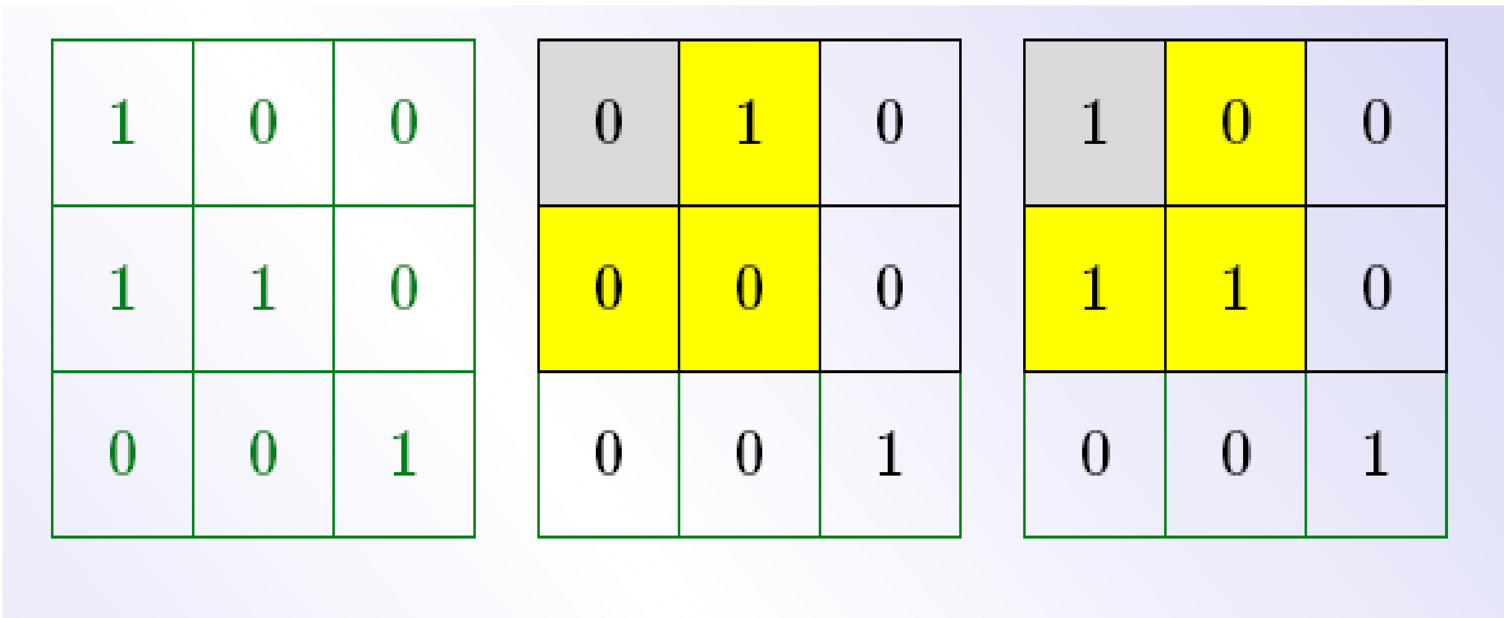
Rules

- When we push the button an odd number of times, it will have the same effect as if we only pushed the button once.
- When we push the button an even number of times, it will have the same effect as if we did not push the button at all.
- The order in which we press the buttons does not affect the final configuration that we're working toward.

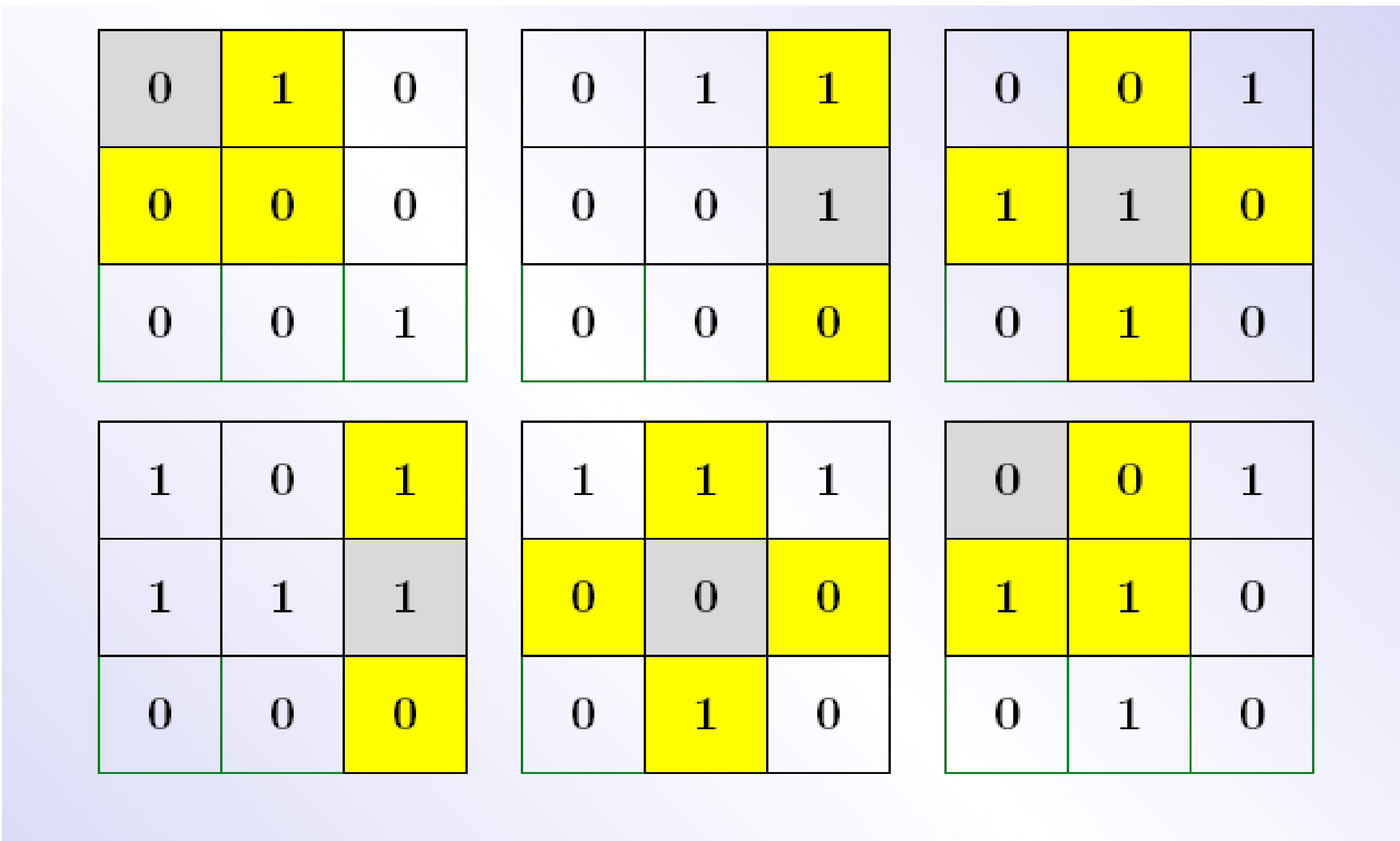
Example (odd)



Example (Even)



Example (Order)



The Goal

1	0	0
1	1	0
0	0	1

1	1	1
1	0	1
1	1	1

Vector Spaces and the Game

For this game we will be working in the binary field in which the only elements are 1 and 0, and where the field operations are addition and multiplication in modulo 2 as shown below:

Addition

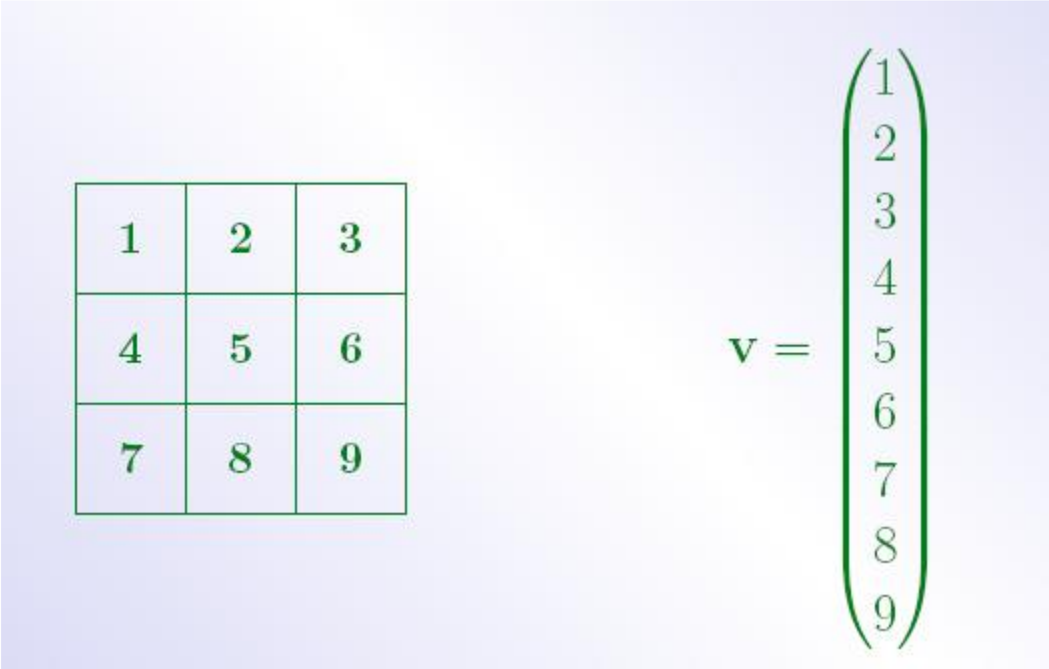
+	0	1
0	0	1
1	1	0

Multiplication

×	0	1
0	0	0
1	0	1

Vector Representation

Think of the array as a vector:



Initial and Winning Vectors

$$\mathbf{v}_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \mathbf{v}_w = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

The game is designed such that for every possible button strike, there is a representative vector u_i that is added to the original vector v_p to produce a new vector $v_{p'}$.

$$\mathbf{v}_p + \mathbf{u}_i = \mathbf{v}_{p'}$$

Example:

$$\mathbf{v}_p + \mathbf{u}_1 = \begin{pmatrix}1\\0\\0\\1\\1\\0\\0\\0\\1\end{pmatrix} + \begin{pmatrix}1\\1\\0\\1\\1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\\1\end{pmatrix} = \mathbf{v}'_p$$

Possibilities

All possible button strikes then, can be represented by 9 different u_i's.

$$\mathbf{u}_1 = (1\ 1\ 0\ 1\ 1\ 0\ 0\ 0\ 0)^T$$

$$\mathbf{u}_2 = (1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0)^T$$

$$\mathbf{u}_3 = (0\ 1\ 1\ 0\ 1\ 1\ 0\ 0\ 0)^T$$

$$\mathbf{u}_4 = (1\ 0\ 0\ 1\ 0\ 0\ 1\ 0\ 0)^T$$

$$\mathbf{u}_5 = (0\ 1\ 0\ 1\ 1\ 1\ 0\ 1\ 0)^T$$

$$\mathbf{u}_6 = (0\ 0\ 1\ 0\ 0\ 1\ 0\ 0\ 1)^T$$

$$\mathbf{u}_7 = (0\ 0\ 0\ 1\ 1\ 0\ 1\ 1\ 0)^T$$

$$\mathbf{u}_8 = (0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1)^T$$

$$\mathbf{u}_9 = (0\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1)^T$$

Getting to the Winning Combination

We can now express the winning configuration, v_w, as a linear combination

$$\mathbf{v}_w = \mathbf{v}_p + s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \ldots + s_9\mathbf{u}_9$$

$$\mathbf{v}_w - \mathbf{v}_p = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \ldots + s_9\mathbf{u}_9$$

$$\mathbf{v}_w - \mathbf{v}_p = \mathbf{v}_w + \mathbf{v}_p$$

Addition mod 2

$$\mathbf{v}_w + \mathbf{v}_p = s_1\mathbf{u}_1 + s_2\mathbf{u}_2 + \ldots + s_9\mathbf{u}_9$$

The Matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

And so $Ax = b$ becomes $A\mathbf{s} = \mathbf{v}_w + \mathbf{v}_p$

What is Needed to Win?

- Find s:

$$As = v_w + v_p$$

$$A^{-1}As = A^{-1}(v_w + v_p)$$

$$s = A^{-1}(v_w + v_p)$$

$$\text{If } \det(A) \neq 0, \text{ then } A^{-1} \text{ exists.}$$

$$|\mathrm{A}| = 1$$

To Win:

$$A^{-1}(\mathbf{v}_w + \mathbf{v}_p) = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \mathbf{s}$$

Some images courtesy of David Arnold at College of the Redwoods.

Also see:

http://www.cut-the-knot.org/Curriculum/Algebra/Merlin.shtml